Injective chromatic number and chromatic number of the square of graphs

Seog-Jin Kim*
Department of Mathematics Education
Konkuk University
Seoul, 143-701, Republic of Korea.

Sang-il Oum†‡
Department of Mathematical Sciences
KAIST
Daejeon, 305-701, Republic of Korea.

February 27, 2009

Abstract

The *injective chromatic number* of a graph G is the minimum number of colors needed in order to color vertices of G so that two vertices with a common neighbor receive distinct colors. We prove that the injective chromatic number of G is at least the half of the chromatic number of G^2, the square of G. This inequality is tight.

An *injective k-coloring* of a graph G is an assignment of at most k colors to the vertices of G such that two vertices sharing a common neighbor must have distinct colors. The *injective chromatic number* $\chi_i(G)$ of a graph G is the minimum k such that G has an injective k-coloring. This notion was

*skim12@konkuk.ac.kr
†sangil@kaist.edu
‡Supported by the SRC Program of Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government (MOST) (No. R11-2007-035-01002-0).

first introduced by Hahn, Kratochvíl, Širáň and Sotteau [1]. We note that an injective coloring need not be a proper coloring; adjacent vertices can have the same color.

The square G^2 of a graph $G = (V, E)$ is the graph on the vertex set V in which two vertices are joined by an edge if their distance in G is at most two. The problem of coloring the square of a graph was started by Wegner [5] in 1977, and has been studied actively. Since a proper injective k-coloring of G is a proper k-coloring of G^2, we have $\chi_i(G) \leq \chi(G^2)$.

Montassier [4] conjectured that $\chi(G^2) \leq 2\chi_i(G)$. In this paper, we prove his conjecture. Consequently

$$\chi_i(G) \leq \chi(G^2) \leq 2\chi_i(G).$$

Hence $\chi(G^2)$ and $\chi_i(G)$ are within factor of 2.

Theorem 1. *For a graph G, we have $\chi(G^2) \leq 2\chi_i(G)$.*

Proof. Let $k = \chi_i(G)$. Then there exists a partition $S_1, S_2, \ldots, S_k$ of the vertex set $V(G)$ of G such that no two vertices in S_i have a common neighbor in G. Then the set of edges with both ends in S_i induces a matching. Therefore we can partition S_i into two sets A_i and B_i, both stable in G. Then A_i and B_i are stable in G^2 too, because no two vertices in A_i or B_i have a common neighbor in G. Then $A_1, A_2, \ldots, A_k, B_1, B_2, \ldots, B_k$ is a partition of $V(G^2)$ into stable sets in G^2. Therefore $\chi(G^2) \leq 2k$. □

Let us explain why Theorem 1 is tight. For an even integer $n \geq 1$, let $G_n = (V, E)$ be a circulant graph on $V = \{0, 1, 2, \ldots, 3n - 1\}$ such that two vertices x, y are adjacent if and only if $x - y \equiv \pm 1 \pmod{3n}$ or $x \equiv y \pmod{3}$.

We claim that

$$\chi(G_n^2) = 2\chi_i(G_n).$$

First of all, it is easy to see that $G_n^2 = K_{3n}$ and therefore $\chi(G_n^2) = 3n$. By Theorem 1, $\chi_i(G_n) \geq 3n/2$. Observe that if two vertices x, y satisfy $x - y \equiv \pm 1 \pmod{3n}$, then x and y have no common neighbors. Therefore in an injective coloring, we can color pairs of consecutive integers with the same color and so $\chi_i(G_n) \leq \lceil 3n/2 \rceil$. We have assumed that n is even and therefore $\chi_i(G_n) = 3n/2$.

Finally let us state an algorithmic consequence of Theorem 1. Hell, Raspaud and Stacho [2] proved that for chordal graphs, the injective chromatic number is α-approximable if and only if the chromatic number of the square is α-approximable. By Theorem 1, we do not have to restrict on chordal graphs.

Corollary 2. *The injective chromatic number is $O(\alpha)$-approximable if and only if the chromatic number of the square is $O(\alpha)$-approximable.*

For example, McCormick [3] constructed an $O(\sqrt{n})$-factor approximation algorithm for computing $\chi(G^2)$ and therefore we conclude that there is an $O(\sqrt{n})$-factor approximation algorithm for $\chi_i(G)$.

References

[1] G. Hahn, J. Kratochvíl, J. Širáň, and D. Sotteau. On the injective chromatic number of graphs. *Discrete Math.*, 256(1-2):179–192, 2002.

[2] P. Hell, A. Raspaud, and J. Stacho. On injective colourings of chordal graphs. *Lecture Notes in Computer Sci.*, 4957:520, 2008.

[3] S. T. McCormick. Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem. *Math. Programming*, 26(2):153–171, 1983.

[4] Montassier. Private communication. 2009.

[5] G. Wegner. Graphs with given diameter and a coloring problem. Technical report, University of Dortmund, 1977.