ON FIRST LAYERS OF $\mathbb{Z}_p$-EXTENSIONS

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Abstract. k denotes a number field. We study the first layers $k_1 \supset k$ of $\mathbb{Z}_p$-extensions of k. k_1 can be described in terms of the norm coherent property over the cyclotomic $\mathbb{Z}_p$-extension of k.

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1. Introduction

We will introduce a theorem which measures the first layers of $\mathbb{Z}_p$-extensions of a number field in terms of the norm coherent property. Firstly, we search possible candidates of the first layers of $\mathbb{Z}_p$-extensions when the ground field contains a primitive pth root of unity. Secondly, we study general case by adding a primitive pth root of unity to the ground field and by shifting $\mathbb{Z}_p$-extensions of the ground field to $\mathbb{Z}_p$-extensions over this bigger field. This is explained at the end of this section. Finally, the first layers of $\mathbb{Z}_p$-extensions are related to Coleman's power series over the p-adic cyclotomic $\mathbb{Z}_p$-extension of the completion of the ground field at a prime lying over p.

The first step is obtained by refining a main result of Bertrandias and Payan in [1] in which they determine the first layers of $\mathbb{Z}/p^n\mathbb{Z}$-extensions of a number field in terms of a certain norm group over the cyclotomic $\mathbb{Z}_p$-extension of the ground field. For a number field k, let Θ_k be the set of all elements α in $k^\times$ such that $k(\alpha^{1/p})$ is a first layer of a $\mathbb{Z}_p$-extension. Let ζ_{p^n} denote a primitive p^nth root of unity in a fixed algebraic closure k^{alg} of k. Let k_∞ be the cyclotomic $\mathbb{Z}_p$-extension of k in k^{alg} with k_n its unique subfield of degree p^n over k. Let $m \geq 1$ denote the maximum number with $\zeta_{p^m} \in k$. By level-shifting over the the cyclotomic $\mathbb{Z}_p$-extension, we may assume in this paper, without loss of generality, that $m = 1$, i.e., $k = k(\zeta_p) \neq k(\zeta_{p^2})$. Then our k_n becomes Bertrandias and Payan's k_{n+1}. For each integer $n \geq 1$, let $\tau(\zeta_{p^n}) = \zeta_{p^n}^{1+p}$. Thus τ defines a topological generator for $G(k_\infty/k)$. Let

$$\Lambda = \mathbb{Z}[[T]] = \varprojlim \mathbb{Z}[G(k_n/k)]$$

be the inverse limit of $\mathbb{Z}[G(k_n/k)]$ for which the generator τ satisfies

$$1 + T = \tau.$$

We will see that Θ_k can be obtained from certain Λ-modules. Let

$$\varprojlim_{n \geq 1} k_n^\times / k^\times (k_n^\times)^{T^2 - Tp}$$

denote the inverse limit of $k_n^\times / k^\times (k_n^\times)^{T^2 - Tp}$ over $n \geq 1$ with respect to the norm maps and let $\varprojlim_{n \geq 1} k_n^\times / k^\times (k_n^\times)^{T - p}$ be defined in the same way. Let π be the natural projection from $\varprojlim_{n\geq 1} k_n^\times/k^\times(k_n^\times)^{T^2-Tp}$ into $k^\times/(k^\times)^p$,

$$\pi((a_n \mathrm{mod}\ k^\times(k_n^\times)^{T^2-Tp})) = N_n(a_n)\mathrm{mod}\ (k^\times)^p$$

where N_n denotes the norm map from k_n to k. We introduce the following conjecture.

Conjecture 1.1. *If k is a number field such that $k = k(\zeta_p) \neq k(\zeta_{p^2})$, then*

$$\Theta_k/(k^\times)^p = \pi(\varprojlim_{n\geq 1} k_n^\times/k^\times(k_n^\times)^{T^2-Tp}).$$

Evidence for the conjecture can be found in the proof of the following theorem.

Theorem 1.2. *If k is a number field such that $k = k(\zeta_p) \neq k(\zeta_{p^2})$, then we have*

$$\pi(\varprojlim_{n\geq 1} k_n^\times/k^\times(k_n^\times)^{T^2-Tp}) \subset \Theta_k/(k^\times)^p \subset \pi(\varprojlim_{n\geq 1} k_n^\times/k^\times(k_n^\times)^{T-p}).$$

We remark here that if k does not contain a primitive pth root of unity, then we have

$$\Theta_k/(k^\times)^p = 0.$$

Next, we move on general case, i.e., when k is an arbitrary number field. If k contains a primitive pth root of unity then the first layers must be of the form $k(\alpha^{1/p})$ for some α in k which was already handled. If k does not contains a primitive pth root of unity then each first layer is not generated by a pth root of an element of k. However, in this case we can find possible candidates of first layers of $\mathbb{Z}_p$-extensions of k by shifting k to $K = k(\zeta_1)$.

$$\begin{tikzcd}[column sep=small] & K_1 = K(\alpha^{1/p}) \arrow[dl, dash] \arrow[dr, dash] & \\ k_1 \arrow[dr, dash] & & K = k(\zeta_1) \arrow[dl, dash] \\ & k & \end{tikzcd}$$

The first layer k_1 of a $\mathbb{Z}_p$-extension k_∞ is linearly disjoint with $k(\zeta_1)$ since the degree is prime to p. In this case, the composite $K_\infty = Kk_\infty$ is a $\mathbb{Z}_p$-extension of K for which we have information about possible candidates of first layers. Since both k_1/k and K/k are abelian with coprime degrees, we have

$$G(K_1/k) \cong \mathbb{Z}/p\mathbb{Z} \times G(k(\zeta_1)/k) \cong \mathbb{Z}/p[k(\zeta_1):k]\mathbb{Z}.$$

The first layer k_1 must be the subfield of K_1 fixed by $G(k(\zeta_1)/k) \cong G(K_1/k_1)$

$$k_1 = K_1^{G(k(\zeta_1)/k)}.$$

Conversely, every first layers of $\mathbb{Z}_p$-extensions of k can be obtained in this way, that is, they are the fixed fields of the first layers of $\mathbb{Z}_p$-extensions of $k(\zeta_1)$ by $G(k(\zeta_1)/k)$.

2. Proof of Theorem 1.2

We start with the following definition.

Definition 2.1. *For an extension field K/k and a group H, we will say K is H-extendable over k if there is an extension field $F \supset K$ such that F/k is Galois and its Galois group $G(F/k)$ is isomorphic to H.*

When K/k is a Galois extension, then K is H-extendable over k if and only if there is a Galois extension $F \supset K$ over k which fits into a short exact sequence

$$0 \longrightarrow G(F/K) \longrightarrow G(F/k) \cong H \longrightarrow G(K/k) \longrightarrow 0$$

For a field F, $F^{\times}$ denotes the multiplicative group of the nonzero elements of F. Following Bertrandias and Payan, we define two subgroups Θ_k and Ψ_k of $k^{\times}/(k^{\times})^p$. Let ζ_n be a primitive nth root of unity in a fixed algebraic closure of k, and let μ_n be the group which is generated by ζ_n. For a number field k, let Θ_k be the set of all elements α in $k^{\times}$ such that $k(\alpha^{1/p})$ is $\mathbb{Z}_p$-extendable. If k does not contain ζ_p, then trivially

$$\Theta_k/(k^{\times})^p = 0.$$

For a field k containing ζ_p, $\Theta_k/(k^{\times})^p$ is a finite dimensional $\mathbb{Z}/p\mathbb{Z}$-vector space. Write s_k for the dimension. In terms of Iwasawa's theory, s_k is the maximum number of independent $\mathbb{Z}_p$-extensions of k. If δ_k denotes the difference between the $\mathbb{Z}$-rank of the global units of k and the $\mathbb{Z}_p$-rank of those in its completion at p, then s_k is equal to $r_2+1+\delta_k$, where r_2 denotes the number of complex embeddings of k. Leopoldt's conjecture is equivalent to $\delta_k = 0$ for a number field k. The conjecture is known to be true for an abelian number field by Brumer. Hence, in abelian number field k, s_k is equal to $r_2 + 1$,

$$\dim_{\mathbb{Z}/p\mathbb{Z}} \Theta_k/(k^{\times})^p = r_2 + 1.$$

In their paper, Bertrandias and Payan studied Θ_k in terms of certain subgroups of the ground field. Among these subgroups are the group of universal norms, the group of p-units. We briefly recall a theorem which played a central role in *loc.cit.* This theorem is a characterizing of elements whose p-th roots generate $\mathbb{Z}/p^n\mathbb{Z}$-extendable fields. Let k_∞ be the cyclotomic $\mathbb{Z}_p$-extension of $k_0 = k = k(\mu_p)$ with k_n its unique subfield of degree p^n over k. Let $m \geq 1$ denote the maximum number with $\zeta_{p^m} \in k$. As mentioned in the introduction, by level-shifting over the the cyclotomic $\mathbb{Z}_p$-extension, we may assume, without loss of generality, that $m = 1$ so that our k_n becomes Bertrandias and Payan's k_{n+1}. Let N_n denote the norm map from k_n to k.

Theorem 2.2 (=Théorème 1 of *loc.cit*)**.** *Let $\alpha \in k^{\times}$. Then $k(\alpha^{1/p})$ is $\mathbb{Z}/p^n\mathbb{Z}$-extendable if and only if $\alpha \in N_{s_n} k_{s_n}^{\times}(k^{\times})^p$, where s_n is $n-m$ if $n \geq m$, and 0 otherwise.*

Based on Theorem 2.2 above, Ψ_k is defined as the set of all $\alpha \in k^{\times}$ such that $k(\alpha^{1/p})$ is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for all n.

Corollary 2.3 (=Corollary of *loc.cit*)**.**

$$\Psi_k = \bigcap_n N_n k_n^{\times}(k^{\times})^p$$

Using these results, it was shown that an extension $k(\alpha^{1/p})$ which is $\mathbb{Z}/p^n\mathbb{Z}$-extendable for all n need not be $\mathbb{Z}_p$-extendable. This was done by providing an example of abelian number field k such that $r_2 + 1 = \dim_{\mathbb{Z}/p\mathbb{Z}} \Theta_k/(k^{\times})^p \neq \dim_{\mathbb{Z}/p\mathbb{Z}} \Psi_k/(k^{\times})^p$, which is due to Serre(cf. *loc.cit*). For a notational convention, write ζ_n for a p^nth root of unity. As defined in the introduction, let τ be a topological generator for $G(k_\infty/k)$ defined to be $\tau(\zeta_n) = \zeta_n^{1+p}$. Let $\tilde{k}_\infty$ denote the inverse limits

$$\tilde{k}_\infty = \varprojlim_{n \geq 1} k_n^{\times}/k^{\times}(k_n^{\times})^{T^2 - Tp}$$

with respect to the norm maps. We have the following exact sequence $0 \to \varprojlim_n k^\times (k_n^\times)^{T^2-Tp} \to \varprojlim_n k_n^\times \to \tilde{k}_\infty$ induced from the short exact sequence $0 \to k^\times (k_n^\times)^{T^2-Tp} \to k_n^\times \to k_n^\times / k^\times (k_n^\times)^{T^2-Tp} \to 0$ by taking inverse limits with respect to the norm maps. Since $\varprojlim_n k^\times (k_n^\times)^{T^2-Tp}$ is trivial, we have

$$\varprojlim_n k_n^\times \hookrightarrow \tilde{k}_\infty.$$

As defined in the introduction, let π be the natural projection from $\tilde{k}_\infty$ into $k^\times/(k^\times)^p$, i.e., $\pi((a_n \mathrm{mod}\ k^\times (k_n^\times)^{T^2-Tp})) = N_n(a_n) \mathrm{mod}\ k^p$. Note that $N_n(a_n)$ is independent of n. Let $\tilde{k}^{\mathrm{coh}}$ denote the image of π inside $k^\times/(k^\times)^p$,

$$\tilde{k}^{\mathrm{coh}} = \pi(\tilde{k}_\infty).$$

Notice that from the inclusion above, we have also the following inclusion

$$\pi(\varprojlim_n k_n^\times) \subset \tilde{k}^{\mathrm{coh}}.$$

We recall the conjecture as was stated in the introduction.

Conjecture 2.4. *If k is a number field such that $k = k(\zeta_p) \neq k(\zeta_{p^2})$, then*

$$\Theta_k/(k^\times)^p = \tilde{k}^{\mathrm{coh}}.$$

Notice that if k does not contain ζ_p, then Θ_k is trivial, i.e., $\Theta_k = (k^\times)^p$. However, $\tilde{k}^{\mathrm{coh}}$ may contain nontrivial elements, for example when $k = \mathbb{Q}$,

$$p \bmod (k^\times)^p = \pi((\zeta_n - 1 \bmod k^\times (k_n^\times)^{T^2-Tp})_n) \in \tilde{k}^{\mathrm{coh}}.$$

The conjecture is true for the following cases. Firstly, if there is only one prime $\mathfrak{p}$ in k lying over p such that the Sylow p-subgroup of the class group of k is generated by the class of $\mathfrak{p}$, then the conjecture is true. Secondly, if $k = \mathbb{Q}(\zeta_p)$ and p is properly irregular, then the conjecture is true. Hence, under Vandiver's conjecture, for all p and $k = \mathbb{Q}(\zeta_p)$, the conjecture is true since for a regular prime, the first case above says the conjecture is true. We do not inquire further into the cases under which the conjecture is true. Instead, we pose the following theorem.

Theorem 2.5. *If k is a number field such that $k = k(\zeta_p) \neq k(\zeta_{p^2})$, then we have*

$$\tilde{k}^{\mathrm{coh}} \subset \Theta_k/(k^\times)^p \subset \pi(\varprojlim_{n \geq 1} k_n^\times / k^\times (k_n^\times)^{T-p}).$$

Proof. Let $\alpha \in \tilde{k}^{\mathrm{coh}}$. For $s \geq n$, let $N_{s,n}$ denote the norm map from k_s to k_n. There is a sequence $\{\alpha_n\}_{n \geq m}$ such that $N_2(\alpha_2) = \alpha$ and $N_{n,n-1}\alpha_{n+1} = \alpha_n a_n \beta_n^{T^2-Tp}$ where $\alpha_n, \beta_n \in k_{n-1}^\times$ and $a_n \in k$. We need the following lemma of *loc.cit.*

Lemma 2.6 (=Lemme 1 of *loc.cit*)**.** *For integers $n > m \geq 1$, there is an integer u_n and a polynomial $f_n'(x)$ in $\mathbb{Z}[x]$ such that*

$$1 + x + \cdots + x^{p^{n-m}-1} = (x - 1 - p) f_n'(x) + (1 + u_n p) p^{n-m}.$$

By replacing variable x by $x - 1 = y$, Lemma 2.6 leads to

$$1 + (y+1) + \cdots + (y+1)^{p^{n-m}-1} = (y - p) f_n'(y+1) + (1 + u_n p) p^{n-m}.$$

Write $f_n'(y+1) = f_n(y)$. When $m = 1$, by plugging in $y = T$, this leads to

$$N_{n-1} = (T - p) f_n(T) + (1 + u_n p) p^{n-1}.$$

Write also $N_{n-1,0}\alpha_n = \alpha\rho_n^{-p}$ where $\alpha_n \in k_{n-1}^\times$ and $\rho_n \in k^\times$, and

$$\nu_n = \alpha_n^{f_n(T)},\ \alpha'_n = \alpha\nu_n^p.$$

The field $L'_n = k_{n-1}((\alpha'_n)^{1/p^n})$ is an abelian extension of k and cyclic extension of degree p^n over k_{n-1}. This fact is contained in the proof of part (b) of THÉORÈME 1 of *loc.cit.* We briefly recall the argument. From Lemma 2.6, we have $\alpha = \rho_n^p(\alpha_n^{1+u_np})^{p^{n-1}}(\alpha_n^{f_n(T)})^{T-p}$ and hence

$$\alpha^p = (\alpha_n^{1+u_np})^{p^n}(\alpha_n^{pf_n(T)}\rho_n^{-p})^{T-p} = (\alpha_n^{1+u_np})^{p^n}\nu_n^{p(T-p)}.$$

It follows that $(\alpha'_n)^{T-p} = \alpha^{-p}\nu_n^{p(T-p)} = (\alpha_n^{1+u_np})^{p^n} \in k_{n-1}^{p^n}$. Let $\tilde{\tau}$ be any extension of τ to L'_n. Since

$$(\tilde{\tau}((\alpha'_n)^{1/p^n}))^{p^n} = \tau(\alpha'_n) = (\alpha'_n)^{1+p}x_n,$$

for some $x_n \in k_n$, $(\tilde{\tau}((\alpha'_n)^{1/p^n})) \in L'_n$ which shows L'_n/k is a Galois extension. Let $\zeta = \sigma((\alpha'_n)^{1/p^n})/(\alpha'_n)^{1/p^n})$. Then $G(L'_n/k)$ is abelian if and only if

$$\tilde{\tau}\sigma((\alpha'_n)^{1/p^n})/\tilde{\tau}((\alpha'_n)^{1/p^n}) = \sigma\tilde{\tau}((\alpha'_n)^{1/p^n})/\tilde{\tau}((\alpha'_n)^{1/p^n}).$$

It follows from $\tau(\sigma((\alpha'_n)^{1/p^n})/(\alpha'_n)^{1/p^n})) = (\sigma((\alpha'_n)^{1/p^n})/(\alpha'_n)^{1/p^n})^{1+p}$ that the condition is equal to the condition that $((\alpha'_n)^{1/p^n})^{1+p}/\tilde{\tau}((\alpha'_n)^{1/p^n})$ is contained in k_{n-1} which is the invariant field by σ. This condition is satisfied since $(\alpha'_n)^{T-p}$ was shown to be a p^nth power of k_{n-1}. This shows that $L'_n = k_{n-1}((\alpha\nu_n^p)^{1/p^n})$ is an abelian extension of k and cyclic extension of degree p^n over k_{n-1}. Next, we claim that

$$L'_n \subset L'_{n+1}.$$

In order to prove the claim, it is enough to show that

$$\nu_{n+1}/\nu_n \in (k_n^\times)^{p^{n-1}}.$$

$$\frac{\nu_{n+1}}{\nu_n} = (\frac{\alpha_{n+1}^{f_{n+1}(T)}}{(N_{n,n-1}\alpha_{n+1}a_n\beta_n^{T^2-Tp})^{f_n(T)}}) = (\frac{\alpha_{n+1}^{f_{n+1}(T)}}{(\alpha_{n+1}^{\sum_{i=1}^p(1+T)^{p^{n-1}i}}a_n\beta_n^{T^2-Tp})^{f_n(T)}}).$$

We show that the first factor

$$(\frac{\alpha_{n+1}^{f_{n+1}(T)}}{(\alpha_{n+1}^{\sum_{i=1}^p(1+T)^{p^{n-1}i}})^{f_n(T)}})$$

and the other factors

$$a_n^{f_n(T)}, \beta_n^{(T^2-Tp)f_n(T)}$$

are p^{n-1}th powers of $k_n^\times$. For the first factor it was shown as in *loc.cit* that

$$(\frac{\alpha_{n+1}^{f_{n+1}(T)}}{(\alpha_{n+1}^{\sum_{i=1}^p(1+T)^{p^{n-1}i}})^{f_n(T)}}) = (\alpha_{n+1}^{\frac{(\sum_{i=1}^p(1+T)^{p^{n-1}i})(1+u_np)-p(1+u_{n+1}p)}{T-p}})^{p^{n-1}}.$$

Since T annihilates a_n, it follows from Lemma 2.6 that

$$a_n^{-pf_n(T)} = a_n^{f_n(T)(T-p)} = a_n^{N_{n-1}-(1+u_np)p^{n-1}} = a_n^{u_np^n}.$$

This shows

$$a_n^{f_n(T)} \in k_n^{p^{n-1}}.$$

Finally, the last factor is a p^{n-1}th powers of $k_n^\times$ since

$$\beta_n^{(T^2-Tp)f_n(T)} = \beta_n^{T(N_{n-1}-(1+u_np)p^{n-1})}$$

from Lemma 2.6. If $k(\alpha^{1/p}) = k_2$ then trivially $\alpha \in \Theta_k$. Otherwise, $k(\alpha^{1/p})$ is linearly disjoint with k_2 over k. Then L'_∞ is a $\mathbb{Z}_p$-extension of k_∞ and L'_∞/k is an abelian extension with $G(L'_\infty/k) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. In this case, it can be shown that $k(\alpha^{1/p})$ is contained in a $\mathbb{Z}_p$-extension of k following the same argument of Lemme 2 of *loc.cit.* More precisely, let σ be a k_∞-automorphism of $k_\infty((\alpha\nu_n^p)^{1/p^n})$. The Galois group decomposes into

$$G(k_\infty((\alpha\nu_n^p)^{1/p^n})/k) \cong \langle\sigma\rangle \oplus \langle\tau\rangle.$$

The field $k(\alpha^{1/p})$ is the fixed field of a subgroup H of $G(k_\infty((\alpha\nu_n^p)^{1/p^n})/k)$ of index p. Hence, H must be of the form $\langle\sigma^p, \tau\sigma^i\rangle$ for some $i, 0 \leq i < p$. Finally, $k(\alpha^{1/p})$ is contained in L_n which is the fixed field of $\langle\tau\sigma^i\rangle$ with $G(L_n/k) \cong G(k_\infty((\alpha\nu_n^p)^{1/p^n})/k)/\langle\tau\sigma^i\rangle \cong \mathbb{Z}/p^n\mathbb{Z}$ and hence, $k(\alpha^{1/p})$ is contained in the $\mathbb{Z}_p$-extension $\varinjlim_n L_n$ of k.

Now, we proceed to the other inclusion. Let $\alpha \in \Theta_k/(k^\times)^p$ so that $k(\alpha^{1/p})$ is a first layer of a $\mathbb{Z}_p$-extension L_∞ of k. There are intermediate fields $L_n \subset L_{n+1}$ and α_i such that $L_{n+1}k_n = k_n(\alpha_{n+1}^{1/p^{n+1}})$ and $L_n k_{n-1} = k_{n-1}(\alpha_n^{1/p^n})$. If L_n and k_n are not linearly disjoint over k, then $L_1 = k(\alpha^{1/p}) = k_1 = k(\zeta_1^{1/p})$. In this case, we can pick an $\alpha_n = \zeta_{n+1}$ so that $(\alpha_n)_n$ is contained in $\varprojlim k_n^\times/k^\times(k_n^\times)^{T-p}$ since $\alpha = \zeta_1 \text{mod } (k^\times)^p$. Thus we may assume that L_n and k_n are linearly disjoint over k.

$$\begin{tikzcd}[column sep=small]
 & & & k_n(\alpha_{n+1}^{1/p^{n+1}}) \arrow[ddll, dash] \arrow[dr, dash] & & & \\
 & & & & k_n(\alpha_n^{1/p^n}) \arrow[dl, dash] \arrow[dr, dash] & & \\
 & L_{n+1} \arrow[dr, dash] & & k_{n-1}(\alpha_n^{1/p^n}) \arrow[dl, dash] \arrow[dr, dash] & & k_n(\alpha^{1/p}) \arrow[dl, dash] \arrow[dr, dash] & \\
 & & L_n \arrow[dr, dash] & & k_{n-1}(\alpha^{1/p}) \arrow[dl, dash] \arrow[dr, dash] & & k_n \arrow[dl, dash] \\
 & & & L_1 = k(\alpha^{1/p}) \arrow[dr, dash] \arrow[rr, phantom, "\perp"] & & k_{n-1} \arrow[dl, dash] & \\
 & & & & k & &
\end{tikzcd}$$

Define ν_i and ξ_i in $k_{i-1}^\times$ as follows. Since $\alpha_i \in \alpha(k_{i-1}^\times)^p$ and $\alpha_{i+1} = \alpha_i(k_i^\times)^{p^i}$, we can write $\alpha_i = \alpha\nu_i^p$ for some $\nu_i \in k_{i-1}^\times$ and

$$\alpha_{n+1} = \alpha_n \beta_{n+1}^{p^n},$$

for some $\beta_{n+1} \in k_n^\times$. Since $k_{i-1}(\alpha_i^{1/p^i})/k$ is abelian for $i = n, n+1$, it follows that $\alpha_i^{T-p} \in (k_{i-1}^\times)^{p^i}$, i.e.,

$$\alpha_i^{T-p} = \xi_i^{-p^i},$$

for some $\xi_i \in k_{i-1}^\times$. Since ξ_{n+1} and β_{n+1} are defined up to multiples of roots of unity, we will choose ξ_{n+1} and β_{n+1} so that they satisfy certain conditions. It follows from the two equations above that

$$\xi_{n+1}^{-p^{n+1}} = \xi_n^{-p^n} \beta_{n+1}^{p^n(T-p)}.$$

Applying the norm map $N_{n,n-1}$ to the equation above, we have

$$N_{n,n-1}(\xi_{n+1}^{-p^{n+1}}) = \xi_n^{-p^{n+1}} N_{n,n-1}\beta_{n+1}^{p^n(T-p)}. \tag{1}$$

Applying the absolute norm N_{n-1}, we have

$$N_{n-1}N_{n,n-1}(\xi_{n+1}^{-p^{n+1}}) = N_{n-1}(\xi_n^{-p^{n+1}} N_{n,n-1}\beta_{n+1}^{-p^{n+1}}).$$

By taking a p^{n+1}th root in the above equation, we have

$$N_{n-1}N_{n,n-1}(\xi_{n+1}) = N_{n-1}(\xi_n N_{n,n-1}\beta_{n+1})\nu,$$

where $\nu \in \mu_{p^{n+1}} \cap k$. Since ν is contained in the image of the norm map, there are $\zeta' \in \mu_n, \delta_n \in k_{n-1}$ such that

$$N_{n,n-1}(\zeta'\xi_{n+1})/\xi_n = N_{n,n-1}\beta_{n+1}\delta_n^T.$$

For a notational convenience, we will denote $\zeta'\xi_{n+1}$ by ξ_{n+1} since ξ_{n+1} is defined up to a multiple of roots of unity. This leads to

$$N_{n,n-1}(\xi_{n+1})/\xi_n = N_{n,n-1}\beta_{n+1}\delta_n^T. \tag{2}$$

It follows from the equations (1) and (2) that

$$N_{n,n-1}\beta_{n+1}^{-p^{n+1}}\delta_n^{-p^{n+1}T} = N_{n,n-1}\beta_{n+1}^{p^n(T-p)}.$$

This leads to

$$N_{n,n-1}\delta_n^{-p^nT} = \delta_n^{-p^{n+1}T} = N_{n,n-1}\beta_{n+1}^{p^nT}.$$

It follows that

$$(N_{n,n-1}\delta_n\beta_{n+1})^T = \nu_1 = \nu_2^T$$

for some $\nu_1 \in \mu_n, \nu_2 \in \mu_{n+1}$. Hence, we have $N_{n,n-1}(\delta_n\beta_{n+1})\nu_2^{-1} \in k^\times$. This shows that $\nu_2 \in \mu_n$, i.e.,

$$N_{n,n-1}(\delta_n\beta_{n+1}) \in k^\times\mu_n. \tag{3}$$

The following lemma is well known. The proof is left to the reader which is elementary.

Lemma 2.7. *Let $(A_n, f_n), (B_n, g_n)$ and (C_n, h_n) be inverse systems. If the short exact sequences*

$$0 \longrightarrow A_n \longrightarrow B_n \xrightarrow{\rho_n} C_n \longrightarrow 0$$

are commute with respect to f_n, g_n and h_n for all n, then we have

$$0 \longrightarrow \varprojlim A_n \longrightarrow \varprojlim B_n \xrightarrow{\varprojlim \rho_n} \varprojlim C_n \longrightarrow \mathrm{Cok}(\varprojlim \rho_n) \longrightarrow 0,$$

where $\mathrm{Cok}(\varprojlim \rho_n)$ is the cokernel of $\varprojlim \rho_n$. If f_n is surjective for all sufficiently large $n \gg 0$, then

$$\mathrm{Cok}(\varprojlim \rho_n) = 0.$$

The equation (2) leads to

$$N_{n,n-1}(\xi_{n+1})/\xi_n = N_{n,n-1}\beta_{n+1}\delta_n^T = N_{n,n-1}(\delta_n\beta_{n+1})\delta_n^{T-p}. \tag{4}$$

The equations (3) and (4) tell us that

$$\Theta_k \subset \pi(\varprojlim k_n^\times / k^\times\mu_{n+1}(k_n^\times)^{T-p}). \tag{5}$$

It follows from Lemma 2.7 that there is a lifting from $\varprojlim k_n^\times/k^\times\mu_{n+1}(k_n^\times)^{T-p}$ to $\varprojlim k_n^\times/k^\times(k_n^\times)^{T-p}$ since the norm maps are surjective over p^nth roots of unity as n varies. Hence, the equation (5) leads to

$$\Theta_k \subset \pi(\varprojlim k_n^\times/k^\times(k_n^\times)^{T-p}).$$

This completes the proof of Theorem 2.5. $\square$

Remark. We write $k = F(\zeta_1)$ with $F/\mathbb{Q}$ is unramified at p and write σ_p for the Frobenius element of p. Then $k_n = F(\zeta_{n+1})$. For each primes $\mathfrak{p}_{n+1}$ in k_{n+1} and $\mathfrak{p}_n$ in k_n with $\mathfrak{p}_{n+1}|\mathfrak{p}_n$ lying over p, we can identify $G(k_{n+1}/k_n)$ with its decomposition group $G(k_{n+1,\mathfrak{p}_{n+1}}/k_{n,\mathfrak{p}_n})$ since each prime lying over p is totally ramified at k_{n+1}/k_n, where $k_{n,\mathfrak{p}_n}$ denotes the completion of k_n at the prime $\mathfrak{p}_n$ and similarly for $k_{n,\mathfrak{p}_{n+1}}$. Hence we have the following commutative diagram

$$\begin{array}{ccccc}
0 & \longrightarrow & k_{n+1} & \longrightarrow & \Pi_{\mathfrak{p}'|p}k^\times_{n+1,\mathfrak{p}'} = (k_{n+1}\otimes\mathbb{Q}_p)^\times \\
 & & \Big\downarrow{\scriptstyle N_{n+1,n}} & & \Big\downarrow{\scriptstyle \Pi_{\mathfrak{p}'|p}N_{n+1,n}} \\
0 & \longrightarrow & k_n & \longrightarrow & \Pi_{\mathfrak{p}|p}k^\times_{n,\mathfrak{p}} = (k_n\otimes\mathbb{Q}_p)^\times
\end{array}$$

where the horizontal arrow denotes the diagonal embedding $k_n^\times \hookrightarrow \Pi_{\mathfrak{p}|p}k^\times_{n,\mathfrak{p}}$. By taking inverse limits, we have

$$\varprojlim k_n^\times \hookrightarrow \varprojlim \Pi_{\mathfrak{p}|p}k^\times_{n,\mathfrak{p}} = \varprojlim(k_n\otimes\mathbb{Q}_p)^\times$$

since the projective limit is left exact functor. Let $\mathcal{O}_F$ denote the ring of integers of F. $\mathcal{O}_F$ is identified with the diagonal embedding via

$$\mathcal{O}_F \hookrightarrow \widehat{\mathcal{O}}_F = \Pi_{\mathfrak{p}|p}\mathcal{O}_{F_\mathfrak{p}} = \mathcal{O}_F\otimes\mathbb{Z}_p$$

where $\mathcal{O}_{F_\mathfrak{p}}$ is the ring of integers of the completion $F_\mathfrak{p}$ of F at a primes $\mathfrak{p}$ lying over p. We need the following theorem of Coleman.

Theorem 2.8 (Theorem 16 and Corollary 17 of [2]). *Let (α_n) be an element of $\varprojlim(k_n\otimes\mathbb{Q}_p)^\times$, where the inverse limit is taken with respect to the norm maps. Then there is a unique power series f in $\widehat{\mathcal{O}}_F((X))$ such that*

$$f(\zeta_{n+1}-1) = \alpha_n^{\sigma_p^n} \text{ and } \mathcal{N}f = f^{\sigma_p}$$

where $\mathcal{N}$ denotes the Coleman's norm operator.

It follows from the inclusion $\pi(\varprojlim_n k_n^\times) \hookrightarrow \tilde{k}^{\mathrm{coh}}$ in the previous argument, Theorem 2.5 and Theorem 2.8 that each element α in $\pi(\varprojlim_n k_n^\times)$ produces a Coleman's power series f such that $k_n(f(\zeta_{n+1}-1)^{1/p^n})$ generates a p^n-extension of k_n which is $\mathbb{Z}_p$-extendable for each n. Conversely, each element f in $(\mathcal{O}_F((X))^\times)^{\mathcal{N}=\sigma_p}$ produces a p^n-extension $k_n(f(\zeta_{n+1}-1)^{1/p^n})$ over k_n which is $\mathbb{Z}_p$-extendable for each n.

References

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