ELLIPTIC UNITS AND CIRCULAR UNITS

SUNGHAN BAE AND SOOGIL SEO

Abstract. Let $K/\mathbb{Q}$ be an abelian extension containing an imaginary quadratic field k. Each intermediate field of the cyclotomic $\mathbb{Z}_p$-extension of K contains both the group of elliptic units and the group of circular units. We explain the relation between the characteristic ideals of the inverse limit of the minus ideal class groups and the inverse limit of the quotients of the circular units by the elliptic units using the Iwasawa main conjectures.

1. Introduction

Let $K/\mathbb{Q}$ be an abelian extension containing an imaginary quadratic field k of odd prime power conductor. Let p be an odd prime which is prime to $[K:k]$ and does not split in k. Let $K_\infty = \bigcup_{n=0}^{\infty} K_n \supset \cdots \supset K_1 \supset K_0$ be the cyclotomic $\mathbb{Z}_p$-extension of $K_0 = K$ with $[K_n : K] = p^n$. For a number field F, let Cl_F denote the ideal class group of F. Then the group of the global units E_n of K_n contains two different subgroups, the circular units $\mathcal{C}_n$ and the elliptic units $\mathcal{E}l_n$, the former is defined in an algebraic way and the latter in an analytic way. Moreover, the group of elliptic units is contained in the group of circular units of K_n. This follows from the results of Gillard(cf. [5] and [9]). Up to a certain constant, the cardinality of the class group of the maximal real subfield of K_n^+ is equal to that of the quotient of E_n by $\mathcal{C}_n$. The group structures of these are not isomorphic in general. When K has a prime power conductor, the cohomologies are known to be Galois isomorphic. Thanks to Gillard(cf. [6]), the cardinality of the p-part of the class group is known to be equal to that of the quotient group of the group of global units by the group of elliptic units when the inertia groups satisfy certain conditions which are satisfied in our case. In general, the cohomologies of the class group Cl_n of K_n and the quotient of E_n by $\mathcal{E}l_n$ are not likely to be isomorphic. As a result, the p-part of the quotient group of the circular units by the elliptic units has the same cardinality as that of the minus class group of K_n. Under the condition of the cyclicity of the minus class group of the first layer, it is known that they have the same module structure. We will show that they have the same characteristic ideals as Iwasawa Λ-modules without the condition of the cyclicity. Let $E_\infty = \varprojlim E_n \otimes \mathbb{Z}_p$, $\mathcal{E}l_\infty = \varprojlim \mathcal{E}l_n \otimes \mathbb{Z}_p$ and $\mathcal{C}_\infty = \varprojlim \mathcal{C}_n \otimes \mathbb{Z}_p$ be the inverse limits of $E_n \otimes \mathbb{Z}_p$, $\mathcal{E}l_n \otimes \mathbb{Z}_p$ and $\mathcal{C}_n \otimes \mathbb{Z}_p$ with respect to the norm maps respectively. Let $\mathrm{Cl}_\infty = \varprojlim \mathrm{Cl}_n \otimes \mathbb{Z}_p$ be the inverse limit of the p-part of the ideal class groups of K_n. Let h_k denote the class number of k.

This work was supported by by the SRC Program of Korea Science and Engineering Foundation (KOSEF) grant funded by the Korea government(R11-2007-035-01001-0)

Theorem 1.1. *Suppose that p does not split completely in K/k. If $p \nmid h_k$, or the Hilbert class field of k is contained in K, then*

$$\mathrm{char}(\mathrm{Cl}_\infty^-) = \mathrm{char}(\mathcal{C}_\infty/\mathcal{E}l_\infty).$$

Let H_n and H_n^+ be the Hilbert class fields of K_n and K_n^+ respectively. Let $H_\infty = \bigcup_n H_n$ and $H_\infty^+ = \bigcup_n H_n^+$.

Corollary 1.2. $\mathrm{char}(\mathcal{C}_\infty/\mathcal{E}l_\infty) = \mathrm{char}(G(H_\infty/K_\infty H_\infty^+))$.

2. Elliptic units and circular units

Following Gillard(cf. [5]), we briefly recall the group of elliptic units. Let $L = \mathbb{Z}w_1 + \mathbb{Z}w_2$ be a lattice in $\mathbb{C}$ with $\mathrm{Im}(w_1/w_2) > 0$. Using Weierstrass' σ function, let

$$\varphi(z, L) = e^{-\eta(z,L)z/2}\sigma(z, L)\Delta^{1/12}(L),$$

where $\eta(z, L)$ is an $\mathbb{R}$-linear form on $\mathbb{C}$(cf. *loc.cit.*) and

$$\Delta^{1/12}(L) = (2\pi i/w_2)q^{1/12}\prod_{n=1}^{\infty}(1 - q^n)^2$$

with $q = e^{2\pi i w_1/w_2}$. For an integral ideal $\mathfrak{f}$ of k, let $k(\mathfrak{f})$ be the ray class field modulo $\mathfrak{f}$ and $Cl(\mathfrak{f}) \cong G(k(\mathfrak{f})/k)$ be its ray class group. For each class $C \in Cl(\mathfrak{f})$, let $(t, \mathfrak{b}) \in k^\times \times Cl(\mathfrak{f})$ be a pair which represents C(cf. *loc.cit.*). Let $\varphi_{\mathfrak{f}}(C) = \varphi^{12f}(t, \mathfrak{b})$, where f denotes the positive generator of $\mathfrak{f} \cap \mathbb{Z}$. Let K/k be a cyclic extension of conductor $\mathfrak{f} \neq 1$. For each $\sigma \in G(K/k)$, let

$$\varphi_K(\sigma) = \prod \varphi_{\mathfrak{f}}(C)$$

where C ranges over all classes in $Cl(\mathfrak{f})$ whose image by the relative reciprocity to K/k is σ. We extend φ_K to the group ring $\mathbb{Z}[G(K/k)]$ by multiplication. Then, φ_K is $G(K/k)$-equivariant map whose image is contained in K. For the unramified case, $\mathfrak{f} = 1$, we refer *loc.cit.* for the details. Let $(-, K/k)$ denote the Artin map over K/k and $N = N_{k/\mathbb{Q}}$ denote the absolute norm map. The subgroup $\tilde{\mathcal{E}l}_K$ of the global units E_K is defined as

$$\tilde{\mathcal{E}l}_K = \{x \in E_K | x^{12fw_{\mathfrak{f}}} \in \varphi_K(J \cap I)\}$$

where $w_{\mathfrak{f}}$ is the number of roots of unity in k congruent to one modulo $\mathfrak{f}$, J the ideal generated by $(\mathfrak{a}, K/k) - N(\mathfrak{a})$ over all ideals of k prime to $12f$, and I is the augmentation ideal of $\mathbb{Z}[G(K/\mathbb{Q})]$. For an abelian extension K/k, the group of elliptic units $\mathcal{E}l_K$ is defined as

$$\mathcal{E}l_K = \prod \tilde{\mathcal{E}l}_{K'}$$

where K' ranges over all subfields of K/k. Notice that in our case, the groups $\tilde{\mathcal{E}l}_{K'}$ coming from unramified fields K' do not have any contributions since K/k has a prime power conductor. Let K^+ denote the maximal real subfield of K. Théorème 2 and Corollaire of *loc.cit.* imply the following result.

If K/k is a ramified extension, then $(\mathcal{E}l_K)^{[K:\mathbb{Q}]^2} \subset \mathcal{C}_K$. Moreover, if all primes which ramify in $K^+/\mathbb{Q}$ also ramify in K/k, then $\mathcal{E}l_K \subset \mu_K\mathcal{C}_{K^+}$.

For the quotient group, there is also an isomorphism

$$(\mathcal{C}_K/\mathcal{E}l_K)\otimes\mathbb{Z}_p \cong \mathbb{Z}_p[K/\mathbb{Q}]/I_K\mathbb{Z}_p[K/\mathbb{Q}],$$

where I_K is the Stickelberger ideal. This is Theorem 4.3 of [11] when K is the pth cyclotomic field, Theorem 5.1 of [8] when K is an intermediate field of the cyclotomic $\mathbb{Z}_p$-extension of an imaginary quadratic field, and Corollaire of [5] for any case satisfying the above containment conditions.

In the following, we will assume that $K/\mathbb{Q}$ is of prime power conductor l^n and $K_\infty = \cup K_n$ is the cyclotomic $\mathbb{Z}_p$-extension. The results above show

$$(\mathcal{C}_n/\mathcal{E}l_n)\otimes\mathbb{Z}_p \cong \mathbb{Z}_p[K_n/\mathbb{Q}]/I_n\mathbb{Z}_p[K_n/\mathbb{Q}]$$

and the following inclusions

$$\mathcal{E}l_n\otimes\mathbb{Z}_p \subset \mathcal{C}_n\otimes\mathbb{Z}_p \subset E_n\otimes\mathbb{Z}_p.$$

For any finite set A, let $\#(A)$ denote the cardinality of A. The class number formula of Gillard shows that

$$\#(\mathrm{Cl}_n\otimes\mathbb{Z}_p) = \#((E_n/\mathcal{E}l_n)\otimes\mathbb{Z}_p)$$

since any primes of K_n not dividing p have ramification degree prime to p and the p-part of $G(K_n/K)$ is cyclic(cf. [6]). Moreover, there is a well known isomorphism for the minus ideal class group of an intermediate field

$$\mathrm{Cl}_n^-\otimes\mathbb{Z}_p \cong \mathbb{Z}_p[K_n/\mathbb{Q}]/I_n\mathbb{Z}_p[K_n/\mathbb{Q}]$$

when $p \nmid [K:k]$ and $\mathrm{Cl}_K^-\otimes\mathbb{Z}_p$ is a cyclic module(cf. [3]). Notice that if K is the pth cyclotomic field and Vandiver's conjecture holds then such isomorphism is also well known. Hence, under certain conditions, we have enough isomorphisms between the minus part of the class group and the quotient group of the group of units modulo the group of elliptic units. With the assumption that $\mathrm{Cl}_K^-\otimes\mathbb{Z}_p$ is a cyclic module, we are led to the equality of two characteristic ideals of the inverse limits of these two objects over the basic $\mathbb{Z}_p$-extension. We will see later that they actually have the same characteristic ideals without the assumption that $\mathrm{Cl}_K^-\otimes\mathbb{Z}_p$ is a cyclic module from the main conjectures of Iwasawa. Since the p-Sylow subgroup of $G(K_n/\mathbb{Q})$ is cyclic Sinnott's class number formula(cf. [16] and [17]) leads to

$$\#(\mathrm{Cl}_n^+\otimes\mathbb{Z}_p) = \#((E_n/\mathcal{C}_n)\otimes\mathbb{Z}_p).$$

Notice that the group structures of these two objects need not be isomorphic in general. This formula leads to

$$\#(\mathrm{Cl}_n^-\otimes\mathbb{Z}_p) = \#((\mathcal{C}_n/\mathcal{E}l_n)\otimes\mathbb{Z}_p).$$

In the cohomological argument to follow, we fix $l = p$. Let f_m be the conductor of K_m. Let $G = G(\mathbb{Q}(\mu_{f_m})/\mathbb{Q})$ and H be any subgroup of G. For the Galois module structures between the cohomology groups of these two objects, we know the following $\mathbb{Z}[G/H]$ isomorphism

$$H^i(H, E_m/\mathcal{C}_m) \cong H^i(H, \mathrm{Cl}_m^+).$$

In [2], this is proved using etale cohomology(cf. [4]). As a result, with the assumption that $\mathrm{Cl}_K^-\otimes\mathbb{Z}_p$ is a cyclic module, we see that

$$H^i(G_{m,n}, \mathrm{Cl}_m^-) \cong H^i(G_{m,n}, \mathcal{C}_m/\mathcal{E}l_m) \text{ and } H^i(G_{m,n}, \mathrm{Cl}_m^+) \cong H^i(G_{m,n}, E_m/\mathcal{C}_m).$$

From the decomposition of the ideal class group $\mathrm{Cl}_m \otimes \mathbb{Z}_p \cong (\mathrm{Cl}_m \otimes \mathbb{Z}_p)^- \oplus (\mathrm{Cl}_m \otimes \mathbb{Z}_p)^+$, we have

$$\begin{aligned} H^i(G_{m,n}, \mathrm{Cl}_m) &\cong H^i(G_{m,n}, \mathrm{Cl}_m^-) \oplus H^i(G_{m,n}, \mathrm{Cl}_m^+) \\ &\cong H^i(G_{m,n}, \mathcal{C}_m/\mathcal{E}l_m) \oplus H^i(G_{m,n}, E_m/\mathcal{C}_m). \end{aligned}$$

In general, it seems very unlikely that $H^i(G_{m,n}, \mathrm{Cl}_m) \cong H^i(G_{m,n}, E_m/\mathcal{E}l_m)$ unless one of the plus or minus parts vanishes. We denote by $E_\infty = \varprojlim E_n \otimes \mathbb{Z}_p$ and $\mathcal{E}l_\infty = \varprojlim \mathcal{E}l_n \otimes \mathbb{Z}_p$ the inverse limits of $E_n \otimes \mathbb{Z}_p$ and $\mathcal{E}l_n \otimes \mathbb{Z}_p$ with respect to the norm maps respectively. We often identify $E_n \otimes \mathbb{Z}_p$ with a subgroup of the principal units $U_n = 1 + \mathfrak{p}_n$ of the local field $K_{n,\mathfrak{p}_n}$, the completion of K_n at $\mathfrak{p}_n$, and similarly we use the same identification for various subgroups. This is possible due to Leopoldt's conjecture. Finally, let $\mathrm{Cl}_\infty = \varprojlim \mathrm{Cl}_n \otimes \mathbb{Z}_p$ be the inverse limit of the p-parts of the ideal class groups of K_n, and similarly, let $\mathcal{C}_\infty = \varprojlim \mathcal{C}_n \otimes \mathbb{Z}_p$ the inverse limits of $\mathcal{C}_n \otimes \mathbb{Z}_p$. Write $\Gamma_n = G(K_\infty/K_n)$. By taking inverse limits to the isomorphism above, we have

$$H^i(\Gamma_n, \mathrm{Cl}_\infty) \cong H^i(\Gamma_n, \mathcal{C}_\infty/\mathcal{E}l_\infty) \oplus H^i(\Gamma_n, E_\infty/\mathcal{C}_\infty).$$

In the study of the characteristic ideals, we let l and p be odd primes so that $K/\mathbb{Q}$ is of prime power conductor l^n and $K_\infty = \cup K_n$ is the cyclotomic $\mathbb{Z}_p$-extension where p does not split completely in $K/\mathbb{Q}$. Notice that under this condition, Gillard's result shows that the p-part of the group of elliptic units is contained in the p-part of the group of circular units of K_n for all n. For $p \nmid [K:k]$, let $\chi : G(K/k) \to (\mathbb{Q}_p^{\mathrm{alg}})^\times$ be a p-adic valued Dirichlet character of $G(K/k)$, where $\mathbb{Q}_p^{\mathrm{alg}}$ is a fixed algebraic closure of $\mathbb{Q}_p$. Let Ξ be the set of p-adic valued Dirichlet characters of $G(K/k)$. Two characters χ and χ' of Ξ are said to be $\mathbb{Q}_p$-conjugate if $\sigma\chi = \chi'$ for some σ in $G(\mathbb{Q}_p^{\mathrm{alg}}/\mathbb{Q}_p)$. This defines an equivalence relation on Ξ. Let $\mathbb{Z}_p[\chi] = \mathbb{Z}_p[\mathrm{image}(\chi)]$ be the $\mathbb{Z}_p[G(K/k)]$-module where $G(K/k)$ acts via $\sigma x = \chi(\sigma)x$ for $\sigma \in G(K/k)$ and $x \in \mathbb{Z}_p[\chi]$. For each χ, we let $e(\chi) = 1/[K:k] \sum_{\sigma \in G(K/k)} \mathrm{Tr}(\chi(\sigma)^{-1})\sigma$ denote the corresponding idempotent with the trace map "Tr" from $\mathbb{Z}_p[\chi]$ to $\mathbb{Z}_p$. Then since $p \nmid [K:k]$, $e(\chi) \in \mathbb{Z}_p[G(K/k]$. For each $\mathbb{Z}[G(K/k)]$-module M of finite type, we let $M(\chi) = e(\chi)(M \otimes \mathbb{Z}_p)$ denote the χ-eigenspace of the p-adic completion $\varprojlim_n M/M^{p^n} = M \otimes \mathbb{Z}_p$ of M. Then we have

$$M \otimes \mathbb{Z}_p \cong \bigoplus_{\chi} M(\chi)$$

where χ runs over all representatives of $\mathbb{Q}_p$-conjugate classes of Ξ. For the characteristic ideals $\mathrm{char}(\mathrm{Cl}_\infty^-)$ and $\mathrm{char}(\mathcal{C}_\infty/\mathcal{E}l_\infty)$, we have the following theorem. Let h_k denote the class number of k.

Theorem 2.1. *Suppose that p does not split completely in K/k. If $p \nmid h_k$, or the Hilbert class field of k is contained in K, then*

$$\mathrm{char}(\mathrm{Cl}_\infty^-) = \mathrm{char}(\mathcal{C}_\infty/\mathcal{E}l_\infty).$$

Proof. By the assumption on p, Gillard's result shows that for each n, $\mathcal{E}l_n \otimes \mathbb{Z}_p$ is contained in $\mathcal{C}_n \otimes \mathbb{Z}_p$. Since the inverse limit is left exact, there exists an inclusion $\mathcal{E}l_\infty \hookrightarrow \mathcal{C}_\infty$ and a natural exact sequence,

$$0 \longrightarrow \mathcal{C}_\infty/\mathcal{E}l_\infty \longrightarrow E_\infty/\mathcal{E}l_\infty \longrightarrow E_\infty/\mathcal{C}_\infty \longrightarrow 0.$$

Hence, the characteristic ideal $\mathrm{char}(E_\infty/\mathcal{E}l_\infty)$ of $E_\infty/\mathcal{E}l_\infty$ splits as follows.

$$\mathrm{char}(E_\infty/\mathcal{E}l_\infty) = \mathrm{char}(\mathcal{C}_\infty/\mathcal{E}l_\infty)\,\mathrm{char}(E_\infty/\mathcal{C}_\infty). \tag{1}$$

For the prime p which satisfies the assumption, Rubin's result(cf. Theorem 4.2 of [13] and [14]) shows one of the main conjectures

$$\mathrm{char}(\mathrm{Cl}_\infty^\chi) = \mathrm{char}((E_\infty/\mathcal{E}l_\infty)^\chi)$$

for any odd character χ of $G(K/k)$ since $\chi(\mathfrak{p}) \neq 1$ and using the class number formula of Gillard. In [13], Rubin proves the above main conjecture under the assumption that K contains the Hilbert class field of k. The assumption can also be replaced by the condition that p is prime to the class number of k(cf. [14]). This leads to

$$\mathrm{char}(\mathrm{Cl}_\infty) = \mathrm{char}((E_\infty/\mathcal{E}l_\infty).$$

Let K_n^+ denote the maximal real subfield of K_n. Let H_n and H_n^+ denote the p-maximal unramified abelian extensions of K_n and K_n^+, respectively. Let H_∞ and H_∞^+ be the maximal unramified abelian extensions of K_∞ and K_∞^+, respectively. We need to prove the following lemma.

Lemma 2.2. $\mathrm{char}(\mathrm{Cl}_\infty^+) = \mathrm{char}(E_\infty/\mathcal{C}_\infty)$.

Proof. Let E_n^+ and $\mathcal{C}_n^+$ denote respectively the global units and the circular units of K_n^+. Since the p-parts $H^0(G(K_n/K_n^+), E_n) \otimes \mathbb{Z}_p$ and $H^0(G(K_n/K_n^+), \mathcal{C}_n) \otimes \mathbb{Z}_p$ of the Tate cohomologies are trivial, we have

$$N_+(E_n) \otimes \mathbb{Z}_p = E_n^+ \otimes \mathbb{Z}_p \ \text{ and } \ N_+(\mathcal{C}_n) \otimes \mathbb{Z}_p = \mathcal{C}_n^+ \otimes \mathbb{Z}_p.$$

Let E_∞^+ and $\mathcal{C}_\infty^+$ denote respectively the inverse limits of $E_n^+ \otimes \mathbb{Z}_p$ and $\mathcal{C}_n^+ \otimes \mathbb{Z}_p$ with respect to the norm maps. This gives the following commutative diagram,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu_p(K_\infty) & \longrightarrow & \mathcal{C}_\infty & \xrightarrow{N_+} & \mathcal{C}_\infty^+ & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu_p(K_\infty) & \longrightarrow & E_\infty & \xrightarrow{N_+} & E_\infty^+ & \longrightarrow & 0
\end{array}$$

Applying the snake lemma to the diagram above, we have

$$E_\infty/\mathcal{C}_\infty \;\cong\; E_\infty^+/\mathcal{C}_\infty^+.$$

From this and the Iwasawa main conjecture of totally real abelian fields(cf. [14]), we obtain $\mathrm{char}(\mathrm{Cl}_\infty^+) = \mathrm{char}(E_\infty/\mathcal{C}_\infty)$, as was claimed in the lemma. □

The decomposition

$$\mathrm{char}(\mathrm{Cl}_\infty) = \mathrm{char}(\mathrm{Cl}_\infty^-)\mathrm{char}(\mathrm{Cl}_\infty^+)$$

induced from $\mathrm{Cl}_\infty = \mathrm{Cl}_\infty^- \oplus \mathrm{Cl}_\infty^+$ and Lemma 2.2 conclude the proof of Theorem 2.1. □

Let H_n and H_n^+ be the Hilbert class fields of K_n and K_n^+ respectively. Let $H_\infty = \bigcup_n H_n$ and $H_\infty^+ = \bigcup_n H_n^+$.

Corollary 2.3. $\mathrm{char}(\mathcal{C}_\infty/\mathcal{E}l_\infty) = \mathrm{char}(G(H_\infty/K_\infty H_\infty^+))$.

Proof. Since K_n/K is totally ramified at a prime lying over p and for all n, we have, by passing through the natural projections, that

$$G(H_\infty/K_\infty) = \varprojlim G(H_n/K_n) \text{ and } G(H_\infty^+/K_\infty^+) = \varprojlim G(H_n^+/K_n^+)$$

$$G(H_\infty/K_\infty H_\infty^+) = \varprojlim G(K_\infty H_\infty^+ H_n/K_\infty H_\infty^+) = \varprojlim G(H_n/K_n H_n^+).$$

We have the following diagram of the fields.

$$\begin{tikzcd}[arrows=dash]
 & K_\infty H_\infty^+ \arrow[r] \arrow[dl] \arrow[dr] & H_\infty \\
K_\infty \arrow[dr] & & H_\infty^+ \arrow[dl] \\
 & K_\infty^+ &
\end{tikzcd}$$

This induces a short exact sequence,

$$0 \longrightarrow G(H_\infty/K_\infty H_\infty^+) \longrightarrow G(H_\infty/K_\infty) \longrightarrow G(H_\infty^+/K_\infty^+) \longrightarrow 0$$

which yields

$$\mathrm{char}(G(H_\infty/K_\infty)) = \mathrm{char}(G(H_\infty/K_\infty H_\infty^+))\,\mathrm{char}(G(H_\infty^+/K_\infty^+)).$$

Let Cl_∞^+ denote the inverse limit of the p-primary parts of the ideal class groups of K_n^+ with respect to the norm maps. It follows from the class field theory that

$$\mathrm{char}(\mathrm{Cl}_\infty) = \mathrm{char}(G(H_\infty/K_\infty H_\infty^+))\,\mathrm{char}(\mathrm{Cl}_\infty^+). \tag{2}$$

Hence we conclude the proof of Corollary 2.3 from Theorem 2.1. $\square$

Remark 1. Let K be the pth cyclotomic field

$$K = \mathbb{Q}(\mu_p).$$

Let χ be a fixed p-adic valued Dirichlet character of $G(K/\mathbb{Q})$. Taking the χ-eigenspaces from (2), we have

$$\mathrm{char}(\mathrm{Cl}_\infty^\chi) = \mathrm{char}(G(H_\infty/K_\infty H_\infty^+)^\chi)\,\mathrm{char}(\mathrm{Cl}_\infty^{+\chi}).$$

Let $K = \mathbb{Q}(\sqrt{-D})$, and let ε be the quadratic character mod D associated K, and let ω be the Teichmuller character. The decomposition above is consistent with the factorization

$$L_p(s, \chi_K) = L_p(s, \chi\varepsilon\omega)\; L_p(1-s, \chi^{-1})$$

of Kubota-Leopoldt p-adic L-series via the main conjecture, where χ denotes a continuous p-adic character of $G(K(\mu_{p^\infty})/\mathbb{Q})$ which is trivial on complex conjugation, and χ_K denotes its restriction to $G(K(\mu_{p^\infty})/K)$(cf. [7]). Note that the relative class number h_K^- of K is essentially a product of the generalized Bernoulli number $\mathbf{B}_{1,\chi}$ over all characters. The factorization of Kubota-Leopoldt p-adic L-series is also related to the factorization of χ-eigenspaces of (1). Indeed, if we plug in $s = 0$ from the factorization of Kubota-Leopoldt p-adic L-series above, we have

$$L_p(0, \chi_K) = L_p(0, \chi\varepsilon\omega)\; L_p(1, \chi^{-1}).$$

Using the same notations of *ibid*, the LHS is a multiple of the p-adic logarithm

$$L_p(0, \chi_K) = -1/3p^r g(\chi^{-1}) \sum \chi(a) \log_p F^+(a)_p$$

of elliptic units, and the first factor of the RHS is a multiple of $\mathbf{B}_{1,\chi}$, and the second factor is a multiple of the p-adic logarithm

$$L_p(1,\chi) = -g(\chi)\sum \chi^{-1}(a)\log_p C^+(a)_p$$

of circular units(cf. page 91 of *ibid*).

Remark 2. We briefly mention the Euler systems between different base fields. As mentioned in the introduction, let $K/\mathbb{Q}$ be an abelian extension containing an imaginary quadratic field k. Let $\mathcal{O}_k$ denote the ring of integers of k. If $u \in \mathcal{E}l_K$, then for every M and $r \geq 0$, there is a truncated Euler system $\alpha \in \mathcal{E}^r_{K/k,M}$ of level r such that $\alpha(\mathcal{O}_k) = u$. This follows immediately from Proposition 1.2 of [13] since the Euler systems are contained in the truncated Euler systems. On the other hand, E_K contains the group of circular units $\mathcal{C}_K$ of Sinnott with $(E_K : \mathcal{C}_K) < \infty$ since $K/\mathbb{Q}$ is abelian. In contrast to the elliptic units, the circular units are defined in an algebraic way. The circular units come from the Euler systems of K over $\mathbb{Q}$ and the elliptic units from that of K over k. The difference between the base fields of the Euler systems results in different notion of the Euler systems because of the difference in their ray class fields. The argument of Euler systems connects the structure of the ideal class group to that of the higher special units(cf. [10] and [15]). We briefly mention their relations in the following. The p-parts $\mathrm{Cl}_K \otimes \mathbb{Z}_p$ and $\mathrm{Cl}_{K^+} \otimes \mathbb{Z}_p$ of the class groups Cl_K of K and Cl_{K^+} of K^+ are related, via the argument of Euler systems, to the direct sums of the quotients of the two consecutive higher special units of each fields as follows.

$$\mathrm{Cl}_K \otimes \mathbb{Z}_p \sim \mathrm{gr}(S^{(p)}_{K/k}) = \bigoplus_{n\geq 1} S^{n-1}_{K/k}/S^n_{K/k}$$

$$\mathrm{Cl}_{K^+} \otimes \mathbb{Z}_p \sim \mathrm{gr}(S^{(p)}_{K^+/\mathbb{Q}}) = \bigoplus_{n\geq 1} S^{n-1}_{K^+/\mathbb{Q}}/S^n_{K^+/\mathbb{Q}}.$$

The circular units C_{K^+} are the higher special units coming from the Euler systems $\mathcal{E}^i_{K^+/\mathbb{Q}}$ over $\mathbb{Q}$, and the elliptic units $\mathcal{E}l_K$ are the higher special units coming from the Euler systems $\mathcal{E}^i_{K/k}$ over k.

$$\mathcal{E}^i_{K/k} \Longrightarrow S^i_{K/k} \otimes \mathbb{Z}_p \overset{i\to\infty}{\Longrightarrow} \mathcal{E}l_K \otimes \mathbb{Z}_p$$

$$\mathcal{E}^i_{K^+/\mathbb{Q}} \Longrightarrow S^i_{K^+/\mathbb{Q}} \otimes \mathbb{Z}_p \overset{i\to\infty}{\Longrightarrow} C_{K^+} \otimes \mathbb{Z}_p.$$

The Euler systems are defined as certain Galois equivariant maps satisfying both the norm conditions and the congruence conditions over the ray class fields of the ground field. Unfortunately, since the ray class fields of different base fields are not norm comparable, the natural norm map does not work satisfactorily between the Euler systems $\mathcal{E}^i_{K/k}$ and $\mathcal{E}^i_{K/\mathbb{Q}}$ even though the norm map works well between the class groups and between various subgroups of the global units. Gillard's proof(cf. [5]) of the containment $\mathcal{E}l_K \otimes \mathbb{Z}_p \subseteq \mathcal{C}_K \otimes \mathbb{Z}_p$ between the p-parts of the elliptic units and the circular units is analytic and does not seem to be comparable with algebraic norm maps.

References

[1] J.-R.Belliard, Sur la structure galoisienne des unités circulaires dans les $\mathbb{Z}_p$-extensions, J. Number Theory 69(1998), 16-49.
[2] D. Burns and S. Seo, On the Galois cohomology of ideal class groups, Arch. Math.(Basel) 89(2007), 536-540.
[3] J. Coates and S. Lichtenbaum, On l-adic zeta functions, Ann. of Math. 98(1973), 498-550.
[4] P. Cornacchia and C. Greither, Fitting ideals of class groups of real fields with prime power conductor. J. Number Theory 73(1998), 459-471.
[5] R. Gillard, Unités elliptiques et unités cyclotomiques, Math. Ann. 243(1979), 181-189.
[6] R. Gillard, Remarques sur les unités cyclotomiques et les unités elliptiques, J. Number Theory 11(1979), 21-48.
[7] B. Gross, On the factorization of p-adic L-series, Invent. Math. 57(1980), 83-95.
[8] T. Itoh, Remark on elliptic units in a $\mathbb{Z}_p$-extension of an imaginary quadratic field, Proc. Amer. Math. Soc. 137(2009), 473-478.
[9] D. Kersey, Modular units inside circular units, Ann. of Math. 112(1980), 361-380.
[10] V. A. Kolyvagin, Euler systems, in: The Grothendieck Festschrift, vol. 2, 435-483, Birkhäuser Verlag, 1990.
[11] D. Kubert and S. Lang, Modular units inside circular units, Bull. Soc. Math. France 107(1979), 161-178.
[12] D. Kubert and S. Lang, Modular units, Grundlehren der Mathematischen Wissenschaften 244. Springer-Verlag, 1981.
[13] K. Rubin, The "main conjectures" of Iwasawa theory for imaginary quadratic fields, Invent. Math. 103(1991), 25-68.
[14] K. Rubin, Euler Systems, Annals of Mathematics Studies, 147, Princeton University Press, 2000.
[15] S. Seo, Truncated Euler Systems, J. reine angew. Math. 614(2008), 53-71.
[16] W. Sinnott, On the Stickelberger ideal and the circular units of a circular field, Ann. of Math. 108(1978), 107-134.
[17] W. Sinnott, On the Stickelberger ideal and the circular units of a circular field, Invent. Math. 62(1980), 181-234.

Department of Mathematics, Yonsei University, 134 Sinchon-Dong, Seodaemun-Gu, Seoul 120-749, Republic of Korea. Email: sgseo@yonsei.ac.kr

Department of Mathematics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Republic of Korea. Email: shbae@math.kaist.ac.kr