On the smooth actions on the Fintushel-Stern's homotopy K3 surfaces

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Abstract

Let $X = X(d_1, d_2, d_3)$ be the Fintushel-Stern's homotopy $K3$ surface, and let G be one of the 11 maximal symplectic $K3$ groups

$$L_2(7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}.$$

In this paper we show that there are no smooth, effective, and pseudo-free actions of G on X. Moreover, we also give a simple proof of the stronger result that the following 6 maximal symplectic $K3$ groups

$$M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, T_{48}$$

cannot act smoothly and effectively on X. These results give some answers to the existence or non-existence questions of smooth actions of the maximal symplectic $K3$ groups on the Fintushel-Stern's homotopy $K3$ surfaces which were initiated by the papers of Chen and Kwasik.

1 Introduction and Main results

The goal of this short paper is to prove some non-existence results of the smooth actions of the maximal symplectic $K3$ groups on the exotic $K3$ surfaces constructed by Fintushel and Stern via the knot surgery method. Such exotic $K3$ surfaces are homeomorphic, but not diffeomorphic to a $K3$ surface. Hence there is a locally linear topological action of the maximal symplectic $K3$ groups on the homotopy $K3$ surfaces, once we fix a homeomorphism between them. As expected, the results of this paper show that a change of smooth structures on a closed oriented smoothable 4-manifold affects the existence of smooth group actions significantly.

In order to explain our main results, let $X(d_1, d_2, d_3)$ be the closed oriented 4-manifold obtained by performing the knot surgery construction simultaneously on three disjoint embedded tori in a Kummer surface representing three distinct homology classes. Here d_1, d_2, d_3 are pairwise relative prime integers with $1 < d_1 < d_2 < d_3$ which are the highest powers of three monic A-polynomials. Then each $X(d_1, d_2, d_3)$ is orientation preserving homeomorphic, but not diffeomorphic to a $K3$ surface, and by the construction and the theorem of Fintushel and Stern, $X(d_1, d_2, d_3)$ admits a symplectic structure, but not a complex structure. (See Section 2 or [2] for more details.)

Recall that a $K3$ surface is a simply connected complex surface with the trivial canonical class. A subgroup of the automorphism group which induces a trivial action on the canonical line bundle is called a *symplectic automorphism group.* A finite group G is called a $K3$ *group* (resp. *symplectic* $K3$ *group*) if G can be realized as a subgroup of the automorphism group (resp. symplectic automorphism group) of a $K3$ surface. It is known in [12] that there exist 11 maximal symplectic $K3$ groups which are all certain subgroups of the Mathieu group M_{23} as follows:

$$L_2(7), A_6, S_5, M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, N_{72}, M_9, T_{48}. \tag{1.1}$$

In this paper we show that all 11 maximal symplectic $K3$ groups on some specific Fintushel-Stern's homotopy $K3$ surfaces cannot act smoothly, effectively, and pseudo-freely. To be precise, our main result is

Theorem 1.1. *Let $X = X(d_1, d_2, d_3)$ be the Fintushel-Stern's homotopy $K3$ surface, and let G be one of the 11 maximal symplectic $K3$ groups as in* (1.1)*. Then there are no smooth, effective, and pseudo-free actions of G on X.*

In fact, in case of the following six maximal $K3$ groups, a version of a much stronger result than Theorem 1.1 was already established in Theorem 1.7 of [2] by Chen and Kwasik. In this paper we will give a much simpler proof of the following stronger result, in Section 3.

Theorem 1.2. *Let $X = X(d_1, d_2, d_3)$ be the Fintushel-Stern's homotopy $K3$ surface, and let G be one of the following 6 maximal symplectic $K3$ groups*

$$M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, T_{48}.$$

Then there are no effective and smooth actions on X.

In the proof of Theorem 1.2, the following theorem will play an important role which is stronger than Theorem 1.7 in [2].

Theorem 1.3. *Let $X = X(d_1, d_2, d_3)$ be the Fintushel-Stern's homotopy $K3$ surface, and let G be a finite group whose commutator $[G, G]$ contains a subgroup isomorphic to $\mathbf{Z}_2^4$ or the quaternionic group Q_8. Then there are no effective and smooth actions on X.*

We organize this paper as follows. In Subsection 2.1, relatively in detail we describe an infinite family of the Fintushel-Stern's homotopy $K3$ surfaces which will be used throughout this paper. Then we give a list of maximal symplectic $K3$ groups together with the information about their commutator groups in Subsection 2.2. Subsection 2.3 is devoted to reviewing the definition of spin number and an important non-existence theorem in [10] which will play a crucial role in the proof of Theorem 3.3. Finally, in Section 3 we provide the proofs of Theorems 1.1, 1.2, and 1.3.

2 Preliminaries

2.1 The Fintushel-Stern's homotopy $K3$ surfaces

The aim of this subsection is to describe in infinite family of closed oriented smooth 4-manifolds constructed by Fintushel and Stern via the knot surgery technique in [6].

To do so, we need to recall the construction of a Kummer surface as in [7]. See also Section 2 in [2]. Let S^1 be the unit circle in $\mathbf{C}$, and let T^4 denote the 4-torus $S^1 \times S^1 \times S^1 \times S^1$. Let $\rho : S^1 \to S^1$ denote the complex conjugation. Then the Kummer surface X can be obtained by resolving 16 singular points in the quotient space T^4/ρ^4. To be precise, we first remove a regular neighborhood of each singular point and then replace it by a regular neighborhood of of an embedded (-2)-sphere. The resulting 4-manifolds for different choices of the gluing map are diffeomorphic to each other. Now, for each $j = 1, 2, 3$, let

$$\pi_j : T^4/\rho^4 \to S^1 \times S^1/\rho^2, \quad (z_0, z_1, z_2, z_3) \mapsto (z_0, z_j)$$

be the projection. Then there is a complex structure J_j on T^4 compatible with the chosen orientation on T^4 such that the projection π_j is holomorphic. Using these three projections π_j $(j = 1, 2, 3)$, we can obtain three C^∞-elliptic fibration $\pi_j : X \to S^2$, and we use three disjoint tori T_j in X for constructing the knot surgery which are the regular fibers $\pi_j^{-1}(p_j, i)$ for some distinct $\pm 1 \neq p_j \in S^1$.

Now assume that d_1, d_2, d_3 are pairwise relative prime integers with $1 < d_1 < d_2 < d_3$, and let $P_j(t)$ be the Laurent polynomial in one variable given by

$$P_j(t) = 1 - (t^{d_j} + t^{-d_j}), \quad j = 1, 2, 3.$$

Then each polynomial $P_j(t)$ is monic and an A-polynomial. Given three A-polynomials $P_1(t)$, $P_2(t)$, and $P_3(t)$, we can perform the knot surgeries simultaneously along the tori T_1, T_2, and T_3 to obtain a simply connected, oriented 4-manifold $X(P_1, P_2, P_3)$ which is orientation-preserving homeomorphic to X and has the Seiberg-Witten invariant

$$SW_{X(P_1,P_2,P_3)} = P_1(t_1)P_2(t_2)P_3(t_3),$$

where $t_j = \exp(2[T_j])$ for $j = 1, 2, 3$. Since all three polynomials P_j are chosen to be monic and an A-polynomial, the 4-manifold $X(P_1, P_2, P_3)$ admits a symplectic structure compatible with the orientation. In this paper, we denote by $X(d_1, d_2, d_3)$ the resulting 4-manifold $X(P_1, P_2, P_3)$.

2.2 Maximal symplectic $K3$ groups

In this section we briefly provide 11 maximal symplectic $K3$ groups and their commutators, for the sake of reader's convenience. See [3] or [13] for more details and notations.

The list of such maximal symplectic $K3$ groups G are:

- $G = L_2(7)$ is a simple group of order $168 = 3 \cdot 7 \cdot 2^4$ and $[G, G] = G$.
- $G = A_6$ is a simple group of order 360 and $[G, G] = G$.
- $G = S_5$ is the symmetric group, $[G, G] = A_5$, and $G/[G, G] = \mathbf{Z}_2$.
- $G = M_{20} = \mathbf{Z}_2^4 \rtimes A_5$ and $[G, G] = G$.
- $G = F_{384} = \mathbf{Z}_4^2 \rtimes S_4$, $[G, G] = \mathbf{Z}_4^2 \rtimes A_4$, and $G/[G, G] = \mathbf{Z}_2$.
- $G = A_{4,4} = \mathbf{Z}_2^4 \rtimes A_{3,3}$, $[G, G] = A_4^2$, and $G/[G, G] = \mathbf{Z}_2$.
- $G = T_{192} = (Q_8 * Q_8) \times_\phi S_3$, $[G, G] = (Q_8 * Q_8) \times_\phi \mathbf{Z}_3$, and $G/[G, G] = \mathbf{Z}_2$.
- $G = H_{192} = \mathbf{Z}_2^4 \rtimes D_{12}$, $[G, G] = \mathbf{Z}_2^4 \rtimes \mathbf{Z}_3$, and $G/[G, G] = \mathbf{Z}_2^2$.
- $G = N_{72} = \mathbf{Z}_3^2 \rtimes D_8$, $[G, G] = A_{3,3}$, and $G/[G, G] = \mathbf{Z}_2^2$.
- $G = M_9 = \mathbf{Z}_3^2 \rtimes Q_8$, $[G, G] = A_{3,3}$, and $G/[G, G] = \mathbf{Z}_2^2$.

- $G = T_{48} = Q_8 \times_\phi S_3$, $[G,G] = T_{24} = Q_8 \times_\phi \mathbf{Z}_3$, and $G/[G,G] = \mathbf{Z}_2$.

The fact that that the following 5 maximal symplectic $K3$ groups

$$L_2(7), A_6, S_5, N_{72}, M_9$$

contains an element of order 3 will be used in the proof of Theorem 3.3 in Section 3.

2.3 Spin numbers

In this subsection we quickly review the definition of spin number which is the index of the Dirac operator associated to a spin structure, and then we recall an important non-existence theorem in [10] which will play a crucial role in the proof of Theorem 3.3. For more details, see Section 3 of [2] and [10].

For the rest of this subsection, let M be a simply connected closed oriented spin 4-manifold, and let G be a cyclic group of an odd prime order p acting on M, unless stated otherwise. Then the action of G on M lifts to the spin structure on M, and the group lifted in such a way, denoted by the same letter G, is isomorphic to G. Let D be the Dirac operator associated to the spin structure. Then for each element $g \in G$, we can define the *Spin number* $\operatorname{Spin}(g, M)$ of g by

$$\operatorname{Spin}(g, M) = \operatorname{tr}(g|_{\ker D}) - \operatorname{tr}(g|_{\operatorname{coker} D}).$$

If we write $\ker D = \oplus_{k=0}^{p-1} V_k^+$ and $\operatorname{coker} D = \oplus_{k=0}^{p-1} V_k^-$, where $V_k^\pm$ is the eigenspace of the lifted element g with eigenvalue $\mu_p^k = e^{\frac{2\pi k i}{p}}$, then we have

$$\operatorname{Spin}(g, M) = \sum_{k=0}^{p-1} d_k \mu_p^k, \tag{2.1}$$

where $d_k = \dim_{\mathbf{C}} V_k^+ - \dim_{\mathbf{C}} V_k^-$ is an integer.

Theorem 2.1. *Assume further that there are only isolated fixed points of G and that each isolated fixed point m is type (a_m, b_m) $(0 < a_m, b_m < p)$ Then the spin number is given by*

(2.2)

$$\operatorname{Spin}(g, M) = \sum_{m \in \{\text{isolated fixed points}\}} -\frac{\epsilon(g, m)}{4} \csc\left(\frac{a_m \pi}{p}\right) \csc\left(\frac{b_m \pi}{p}\right),$$

where the sign $\epsilon(g, m) = \pm 1$ depends on the fixed point m and the lifting of the action of g to the spin structure.

In general, it is very delicate to determine the sign $\epsilon(g, m)$ in the above formula (2.2). However, in the case that p is an odd prime, it can be determined explicitly as in [8], [11], and [2].

If M is a homotopy $K3$ surface, in the paper [10] we have given some obstruction to the existence of a periodic diffeomorphism of odd prime order acting trivially on the self-dual part $H^2_+(M;\mathbf{R})$ in terms of the rationality and negativity of the spin number. The following theorem (Theorem 1.3 in [10]) will also play an important role in the proof of Theorem 3.3.

Theorem 2.2. *Let M be a homotopy $K3$ surface, and let $g : M \to M$ be a periodic diffeomorphism of odd prime order p. Assume that the spin number* $\mathrm{Spin}(g, X)$ *is both rational and negative. Then g cannot act trivially on the self-dual part $H^2_+(M;\mathbf{R})$ of the second cohomology group.*

3 Proofs of Theorems 1.1, 1.2 and 1.3

The aim of this section is to give proofs of Theorems 1.1, 1.2 and 1.3. We first begin with the following theorem whose statement is much stronger than Theorem 1.7 in [2], but whose proof is much simpler.

Theorem 3.1. *Let $X = X(d_1, d_2, d_3)$ be the Fintushel-Stern's homotopy $K3$ surface, and let G be a finite group whose commutator $[G, G]$ contains a subgroup isomorphic to $\mathbf{Z}_2^4$ or the quaternionic group Q_8. Then there are no effective and smooth actions on X.*

Proof. In case that the commutator group $[G, G]$ contains a subgroup isomorphic to $\mathbf{Z}_2^4$, it has already been shown in Theorem 1.7 of [2] that G cannot act on X smoothly and effectively. So it suffices to consider the remaining quaternionic group case.

To do so, let g be an element of order 4 in Q_8. Since g lies in the commutator $[G, G]$ by assumption and $g_*([T_i]) = \pm[T_i]$ for all $i = 1, 2, 3$, g fixes three homology classes $[T_1], [T_2]$, and $[T_3]$. Thus it follows from Lemma 4.1 in [2] that $b_2^+(X/g) = 3$, which implies that $b_2^+(X/g^2) = 3$. Since X is simply connected and spin, g^2 should be an involution of even type with 8 isolated fixed points by the main result in [1] (see also [9]). Thus g has at most 8 isolated fixed points.

Let s_+ (resp. s_-) be the number of isolated fixed points whose weights of the local representation are $(1, 3)$ (resp. $(1, 1)$ or $(3, 3)$). Then it was shown in Lemma 5.3 of [2] that $s_+ = 4$ and $s_- = 0, 2, 4$. Hence the number of isolated fixed points of g is in fact either $4, 6$, or 8.

Recall that the quaternionic group Q_8 is given by

$$Q_8 = \{i, j, k \mid i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j\}.$$

Then it is easy to see that it suffices to consider only the following possibilities for the triple (p, q, r) consisting of the numbers of isolated fixed points for i, j, and k, respectively:

$$\begin{array}{lll} (4,4,4), & (4,4,6), & (4,4,8) \\ (6,6,6), & (6,6,4), & (6,6,8) \\ (8,8,8), & (8,8,4), & (8,8,6). \end{array}$$

Observe that if an isolated point is fixed by i and j (or j and k), then the point is also fixed by k (or i). So it is easy to see from a simple combinatorial argument that those cases with $(p, q, r) = (6,6,8)$, $(8,8,4)$, $(8,8,6)$ do not occur.

Next we want to reduce the above list further to the case $(4,4,4)$. To do so, note first that if the triple (p, q, r) in the above list is different from $(4,4,4)$, then there exists an isolated fixed point of i, j, or k with nonzero s_-. So, we assume without loss of generality that there exists an isolated fixed point P of k whose type is, say, $(1,1)$. In fact, they are all the cases in the *reduced* list except (p, q, r) equal to $(4,4,4)$ or $(6,6,4)$. Since $jkj^{-1} = k^{-1}$ and the type of k coincides with that of its conjugate jkj^{-1}, the type of k^{-1} at the point P is $(1,1)$ by its choice. But this gives rise to a contradiction, since the type of k^{-1} at P would become $(3,3)$. Therefore we are now reduced to the case that the triple (p, q, r) is $(4,4,4)$. It is, however, already shown in the proof of Theorem 1.7 in [2] that this case does not occur, either. Hence we have completed the proof of Theorem 3.1. $\square$

Now it is immediate to obtain the following corollary, since the commutator group of all the maximal symplectic $K3$ groups below contains a subgroup isomorphic to $\mathbf{Z}_2^4$ or Q_8.

Corollary 3.2. *Let $X = X(d_1, d_2, d_3)$ be the Fintushel-Stern's homotopy $K3$ surface, and let G be one of the following 6 maximal symplectic $K3$ groups*

$$M_{20}, F_{384}, A_{4,4}, T_{192}, H_{192}, T_{48}.$$

Then there are no effective and smooth actions of G on X.

Next we give a proof of Theorem 1.1. In view of Corollary 3.2 above, it suffices to prove the following theorem.

Theorem 3.3. *Let $X = X(d_1, d_2, d_3)$ be the Fintushel-Stern's homotopy $K3$ surface, and let G be one of the following 5 maximal symplectic $K3$ groups*

$$L_2(7), A_6, S_5, N_{72}, M_9. \tag{3.1}$$

Then G cannot act smoothly, effectively, and pseudo-freely on X.

Proof. Note that all the 5 maximal symplectic $K3$ groups in (3.1) contains an element g of order 3 in $[G, G] \subset G$. Since g is of odd order or lies in the commutator $[G, G]$, all the homology classes $[T_i]$ $(i = 1, 2, 3)$ are fixed by g. Hence it follows from Lemma 4.1 of [2] that $b_2^+(X/g) = 3$.

Now suppose that there exists a smooth, effective and pseudo-free action of G on X. Let s_+ (resp. s_-) be the number of isolated fixed points of H of type $(1, 2)$ or $(2, 1)$ (resp. $(1, 1)$ or $(2, 2)$). It is known in [11] that we have the following possibilities for various topological data, as follows.

$\lvert X^{\langle g\rangle}\rvert$	s_+	s_-	$b_2(X/\langle g\rangle)$	$b_2^+(X/\langle g\rangle)$	$b_2^-(X/\langle g\rangle)$	$\sigma(X/\langle g\rangle)$
6	6	0	10	3	7	-4
12	0	12	14	3	11	-8

Here $\lvert X^{\langle g\rangle}\rvert$ and $\sigma(X/\langle g\rangle)$ mean the number of the isolated fixed points of $\langle g\rangle$ and the signature of $X/\langle g\rangle$, respectively. Note also that the results of Liu and Nakamura in [11] have been established for locally linear topological actions on a $K3$ surface, but not a homotopy $K3$ surface. However, their results also work for homotopy $K3$ surfaces.

Recall now that for prime integer p the first Betti number of the fixed point set with $\mathbf{Z}_p$-coefficients is equal to the number of copies of $\mathbf{Z}_p$ representations of cyclotomic type in $H_2(X)$ (see Proposition 3.1 in [2] or Proposition 2.4 in [4]). Since in our case the action is pseudo-free, there are no copies of $\mathbf{Z}_p$-representations of cyclotomic type in $H_2(X)$. In that case, it follows from Proposition 4.6 in [2] that for $p = 3, 5, 7$ the intersection form on $H_2(X, \mathbf{Z})$ can be decomposed as $3H \oplus 2E_8$ in such a way that each summand H or E_8 is invariant under the $\mathbf{Z}_p$ action. In particular, the action of $\mathbf{Z}_3 = \langle g\rangle$ on each summand H is trivial, since the action of $\langle g\rangle$ is pseudo-free. Hence the dimension of the invariant subspace of E_8 under the action of $\langle g\rangle$ is either 2 or 4 from the above table. It is important to note that this property is very special in that it applies only to the Fintushel-Stern's homotopy $K3$ surface.

Next we show that the dimension of the invariant subspace of E_8 under the action of $\langle g\rangle$ should be 4. To do so, recall that $\mathrm{Aut}(E_8 \oplus E_8)$ is a semi-direct product of $\mathrm{Aut}(E_8) \times \mathrm{Aut}(E_8)$ by $\mathbf{Z}_2$. Thus there exists a representation $\Theta = (\Theta_1, \Theta_2)$ given by

$$\Theta : \langle g \rangle \to \mathrm{Aut}(E_8) \times \mathrm{Aut}(E_8) \subset \mathrm{Aut}(E_8 \oplus E_8).$$

Then it follows from the Lefschetz fixed point theorem (e.g., Theorem 3.4 of [2]) and the above table that we have

$$\mathrm{tr}(\Theta_1(g)) + \mathrm{tr}(\Theta_2(g)) = -2 \text{ or } 4. \tag{3.2}$$

It is well-known (e.g., Lemma 4.5 in [2] or [5]) that the integral representation of $\mathbf{Z}_3$ induced from $\mathbf{Z}_3$ in $\mathrm{Aut}(E_8)$ is either

$$\mathbf{Z}[\mathbf{Z}_3] \oplus \mathbf{Z}^5,\ \ \mathbf{Z}[\mathbf{Z}_3]^2 \oplus \mathbf{Z}^2,\ \ \mathbf{Z}[\mathbf{Z}_3] \oplus \mathbf{Z}[\mu_3]^2 \oplus \mathbf{Z},\ \text{ or } \mathbf{Z}[\mu_3]^4.$$

Since the action of $\mathbf{Z}_3 = \langle g \rangle$ is assumed to be pseudo-free, the last two cases containing the copies of cyclotomic representations do not occur. Notice that the group $\mathbf{Z}_3 = \langle g \rangle$ whose integral representation is $\mathbf{Z}[\mathbf{Z}_3] \oplus \mathbf{Z}^5$ or $\mathbf{Z}[\mathbf{Z}_3]^2 \oplus \mathbf{Z}^2$ is conjugate to the subgroup generated by the order three cyclic permutation or the product of two odd three cyclic permutations. Thus the trace $\mathrm{tr}(\Theta_i(g))$ $(i = 1, 2)$ is either 5 or 2. Since the sum $\mathrm{tr}(\Theta_1(g)) + \mathrm{tr}(\Theta_2(g))$ of two traces is either -2 or 4 by (3.2), we have only one possibility $\mathrm{tr}(\Theta_i(g)) = 2$ $(i = 1, 2)$. It is the case that the integral representation of $\mathbf{Z}_3$ induced from $\mathbf{Z}_3$ in $\mathrm{Aut}(E_8)$ is $\mathbf{Z}[\mathbf{Z}_3]^2 \oplus \mathbf{Z}^2$. So the dimension of the invariant subspace of E_8 under the action of $\langle g \rangle$ should be 4. Now, recall that this case happens only when $|X^{\langle g \rangle}| = 12$, $s_+ = 0$ and $s_- = 12$ from the above discussion.

In order to finish the proof, we finally compute the spin number (e.g, see (2.1) and Theorem 2.1 of the present paper or Theorem 3.7 in [2])

$$\begin{aligned} \mathrm{Spin}(g, X) &= \sum_{k=0}^{2} d_k \mu_3^k \\ &= \sum_{m \in \{\text{isolated fixed points}\}} -\frac{\epsilon(g,m)}{4} \csc\left(\frac{a_m \pi}{3}\right) \csc\left(\frac{b_m \pi}{3}\right), \end{aligned} \tag{3.3}$$

where all the d_k's $(k = 0, 1, 2)$ are integers, $\mu_3 = e^{\frac{2\pi i}{3}}$, and each isolated fixed point m is assumed to be of type (a_m, b_m) $(0 < a_m, b_m < 3)$. For this computation we first need to determine the sign $\epsilon(g, m)$ of each isolated fixed point, and in our case the sign turns out to be equal to $+1$ by an argument of Liu and Nakamura in Section 3 of [11]. Alternatively, we can also see this as follows: if the sign $\epsilon(g,m)$ of each isolated fixed point is -1, then the spin number

$$\mathrm{Spin}(g,X)=\sum_{m\in\{\text{isolated fixed points}\}}-\frac{\epsilon(g,m)}{4}\csc^2\left(\frac{\pi}{3}\right)=4.$$

Thus it follows from (3.3) that we have $d_0-\frac{1}{2}(d_1+d_2)=4$ and $d_1=d_2$. Since $d_0+d_1+d_2=-\frac{\sigma(X)}{8}=2$, we have $d_0+2d_1=2$ and so we can obtain $3d_0=10$. This does not make sense.

Hence we can conclude that the spin number $\mathrm{Spin}(g,X)$ is -4. In particular, this implies that the spin number is both rational and negative. But then it follows from Theorem 2.2 (or Theorem 1.3 of [10]) that g cannot act trivially on the self-dual part $H^2_+(X,\mathbf{R})$, which is a contradiction to $b_2^+(X/g)=3$. This completes the proof of Theorem 3.3. $\square$

As a byproduct, the proof of Theorem 3.3 also shows that there are no smooth, effective, and pseudo-free action of $\mathbf{Z}_3$ on a homotopy $K3$ surface, not just the Fintushel-Stern's homotopy $K3$ surface, which satisfies the conditions $|X^{\langle g\rangle}|=12$, $s_+=0$, $s_-=12$, $b_2(X/\langle g\rangle)=14$, $b_2^+(X/\langle g\rangle)=3$, and $b_2^-(X/\langle g\rangle)=11$. This enables us to eliminate the so-called A_2 case in the list of Theorem 1.2 of [11] or the second case of the above table. However, in the first case of the above table the sign $\epsilon(g,m)$ of each isolated fixed point is -1 and so the spin number is positive. Thus we cannot eliminate this case, and it is indeed realized by a smooth and pseudo-free action of $\mathbf{Z}_3$ on the Fermat quartic surface. Hence we have the following theorem.

Theorem 3.4. *Let X be a homotopy $K3$ surface, and let $g:X\to X$ be a periodic diffeomorphism of order 3 acting pseudo-freely on X. If g acts trivially on the self-dual part $H^2_+(M;\mathbf{R})$ of the second cohomology group, then the following holds:*

$$|X^{\langle g\rangle}|=6,\ s_+=6,\ s_-=0,\ b_2(X/\langle g\rangle)=10.$$

Finally, Theorem 1.1 follows immediately from Theorem 3.3 and Corollary 3.2.

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