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Algebraic Montgomery-Yang Problem: the non-cyclic case

DongSeon Hwang · JongHae Keum *

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Abstract. Montgomery-Yang problem predicts that every pseudofree circle action on the 5-dimensional sphere has at most 3 non-free orbits. Using a certain one-to-one correspondence, Kollár formulated the algebraic version of the Montgomery-Yang problem: every projective surface S with quotient singularities such that the second Betti number $b_2(S) = 1$ has at most 3 singular points if its smooth locus S^0 is simply connected.

We prove the conjecture under the assumption that S has at least one non-cyclic singularity. In the course of the proof, we classify projective surfaces S with quotient singularities such that (i) $b_2(S) = 1$, (ii) $H_1(S^0, \mathbb{Z}) = 0$, and (iii) S has 4 or more singular points, not all cyclic, and prove that all such surfaces have $\pi_1(S^0) \cong \mathfrak{A}_5$, the icosahedral group.

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1. Introduction

A pseudofree $\mathbb{S}^1$-action on a sphere $\mathbb{S}^{2k-1}$ is a smooth $\mathbb{S}^1$-action which is free except for finitely many non-free orbits (whose isotropy types $\mathbb{Z}_{m_1}, \ldots, \mathbb{Z}_{m_n}$ have pairwise relatively prime orders).

For $k = 2$ Seifert [18] showed that such an action must be linear and hence has at most two non-free orbits. In the contrast to this, for $k = 4$ Montgomery and Yang [15] showed that given any pairwise relatively prime collection of positive integers $m_1, \ldots, m_n$, there is a pseudofree $\mathbb{S}^1$-action on homotopy 7-sphere whose non-free orbits have exactly those orders. Petrie [16] proved sim-

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DONGSEON HWANG
Department of Mathematical Sciences, KAIST, Daejon, Korea,
e-mail: `themiso@kaist.ac.kr`
Current Address:
School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea,
e-mail: `dshwang@kias.re.kr`

JONGHAE KEUM
School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea,
e-mail: `jhkeum@kias.re.kr`

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ilar results in all higher odd dimensions. This led Fintushel and Stern to formulate the following problem:

Conjecture 1 ([3]). (Montgomery-Yang Problem)
Let

$$\mathbb{S}^1 \times \mathbb{S}^5 \to \mathbb{S}^5$$

be a pseudofree $\mathbb{S}^1$*-action. Then it has at most* 3 *non-free orbits.*

The problem has remained unsolved since its formulation.

Pseudofree $\mathbb{S}^1$-actions on 5-manifolds L have been studied in terms of the 4-dimensional quotient orbifold $L/\mathbb{S}^1$ (see e.g., [2], [3]). A manifold is called a *rational homology sphere* if it has the same $\mathbb{Q}$-homology groups with a sphere, i.e., it has the same Betti numbers with a sphere. The following one-to-one correspondence was known to Montgomery, Yang, Fintushel and Stern, and recently observed by Kollár ([11], [12]):

Theorem 1 (cf. [11], [12]). *There is a one-to-one correspondence between:*

1. *Pseudofree* $\mathbb{S}^1$*-actions on* 5 *dimensional rational homology spheres* L *with* $H_1(L, \mathbb{Z}) = 0$.
2. *Smooth, compact* 4*-manifolds* M *with boundary such that*
   (a) $\partial M = \cup_i L_i$ *is a disjoint union of lens spaces* $L_i = \mathbb{S}^3/\mathbb{Z}_{m_i}$*,*
   (b) the m_i *are relatively prime to each other,*
   (c) $H_1(M, \mathbb{Z}) = 0$ *and* $H_2(M, \mathbb{Z}) \cong \mathbb{Z}$*.*

Furthermore, L *is diffeomorphic to* $\mathbb{S}^5$ *iff* $\pi_1(M) = 1$.

We recall that a normal projective surface with the same Betti numbers with $\mathbb{CP}^2$ is called a *rational homology projective plane* or a $\mathbb{Q}$*-homology projective plane* or a $\mathbb{Q}$*-homology* $\mathbb{CP}^2$. When a normal projective surface S has quotient singularities only, S is a $\mathbb{Q}$-homology $\mathbb{CP}^2$ if the second Betti number $b_2(S) = 1$.

It is known that a $\mathbb{Q}$-homology projective plane with quotient singularities has at most 5 singular points (cf. [4] Corollary 3.4). Recently, the authors have classified $\mathbb{Q}$-homology projective planes with 5 quotient singularities ([4], also see [8]).

Using the one-to-one correspondence of Theorem 1.2, Kollár formulated the algebraic version of the Montgomery-Yang problem as follows:

Conjecture 2 ([12]). (Algebraic Montgomery-Yang Problem)
Let S *be a* $\mathbb{Q}$*-homology projective plane with quotient singularities. Assume that* $S^0 := S \backslash Sing(S)$ *is simply connected. Then* S *has at most* 3 *singular points.*

In this paper, we verify the conjecture when S has at least one non-cyclic singularity. More precisely, we prove the following:

Theorem 2. *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities such that $\pi_1(S^0) = \{1\}$. Assume that S has at least one non-cyclic singularity. Then $|Sing(S)| \leq 3$.*

We note that the condition $\pi_1(S^0) = \{1\}$ cannot be replaced by the weaker condition $H_1(S^0, \mathbb{Z}) = 0$. There are infinitely many examples of $\mathbb{Q}$-homology projective planes with exactly 4 quotient singularities, where three of them are cyclic and one of them is non-cyclic, such that $H_1(S^0, \mathbb{Z}) = 0$ but $\pi_1(S^0) \neq \{1\}$ ([1] or [12], Example 31). These examples are the global quotients

$$S_{I_m} := \mathbb{CP}^2/I_m = (\mathbb{CP}^2/Z)/\mathfrak{A}_5,$$

where $I_m \subset GL(2, \mathbb{C})$ is the group of order $120m$ in Brieskorn's list (see Table 1), an extension of the icosahedral group $\mathfrak{A}_5 \subset PGL(2, \mathbb{C})$ by the cyclic group $Z \cong \mathbb{Z}_{2m}$, and the action of I_m on $\mathbb{CP}^2$ is induced from the natural action on $\mathbb{C}^2$. We call S_{I_m} a *Brieskorn quotient*.

On the other hand, it follows from the orbifold Bogomolov-Miyaoka-Yau inequality that every $\mathbb{Q}$-homology projective plane S with quotient singularities such that $H_1(S^0, \mathbb{Z}) = 0$ has at most 4 singular points(cf. [12], [4]). Therefore, to prove Theorem 2, it is enough to classify $\mathbb{Q}$-homology projective planes S with 4 quotient singularities, not all cyclic, such that $H_1(S^0, \mathbb{Z}) = 0$. It turns out that such a surface is deformation equivalent to a Brieskorn quotient.

Theorem 3. *Let S be a $\mathbb{Q}$-homology projective plane with 4 quotient singularities, not all cyclic, such that $H_1(S^0, \mathbb{Z}) = 0$. Then the following hold true.*

1. *S has 3 cyclic singularities of type $\mathbb{C}^2/\mathbb{Z}_2$, $\mathbb{C}^2/\mathbb{Z}_3$, $\mathbb{C}^2/\mathbb{Z}_5$, and one non-cyclic singularity of type $\mathbb{C}^2/I_m$, where $I_m \subset GL(2, \mathbb{C})$ is the $2m$-ary icosahedral group of order $120m$ (in Brieskorn's notation). Furthermore, the 3 cyclic singularities are of type $\frac{1}{2}(1, 1)$, $\frac{1}{3}(1, \alpha)$, $\frac{1}{5}(1, \beta)$, if the 3 branches of the dual graph of the non-cyclic singularity are of type $\frac{1}{2}(1, 1)$, $\frac{1}{3}(1, 3 - \alpha)$, $\frac{1}{5}(1, 5 - \beta)$ (see Table 4).*
2. *$-K_S$ is ample.*
3. *The minimal resolution of S can be obtained by starting with a minimal rational ruled surface and blowing up inside 3 of the fibres, i.e. the blowing up starts at three centers, one on each of the 3 fibres.*
4. *S^0 is deformation equivalent to $(\mathbb{CP}^2/I_m)^0$, where I_m is determined by the non-cyclic singularity of S and its action on $\mathbb{CP}^2$ is induced by the natural action on $\mathbb{C}^2$. The deformation space has dimension 2.*
5. *$\pi_1(S^0) \cong \mathfrak{A}_5$, the alternating group of order 60.*

In the proof, we use the orbifold Bogomolov-Miyaoka-Yau inequality (Theorem 4 and 5) and a detailed computation for (-1)-curves on the minimal resolution S' of S. The latter idea was used in [7].

In the cyclic case (where S has cyclic singularities only), Conjecture 1.3 has been confirmed in a separate paper [5] unless S is a rational surface with K_S ample.

Remark 1. Consider a Brieskorn quotient $S_{I_m} := \mathbb{CP}^2/I_m = (\mathbb{CP}^2/Z)/\mathfrak{A}_5$. The cone $\mathbb{CP}^2/Z$ is the closure of the $\mathfrak{A}_5$-universal cover of $S^0_{I_m}$. Note that the cone has no deformation. Thus the deformation of $S^0_{I_m}$ must correspond to a deformation of the I_m-action on $\mathbb{CP}^2$. This was pointed out to us by János Kollár. It is an interesting problem to describe explicitly such a deformation.

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

2. Algebraic surfaces with quotient singularities

2.1. Classification of quotient surface singularities

A singularity p of a normal surface S is called a quotient singularity if the germ is locally analytically isomorphic to $(\mathbb{C}^2/G, O)$ for some nontrivial finite subgroup G of $GL_2(\mathbb{C})$ without quasi-reflections. Brieskorn classified such finite subgroups of $GL(2,\mathbb{C})$ [Bri]. Table 1 summarizes the result. Here we only explain the notation for dual graph.

$$\begin{aligned} &< q, q_1 > && := \text{ the dual graph of the singularity of type } \frac{1}{q}(1, q_1) \\ &< b; s_1, t_1; s_2, t_2; s_3, t_3 > && := \text{ the tree of the form} \end{aligned}$$

$$\begin{array}{ccccc} & & < s_2, t_2 > & & \\ & & | & & \\ < s_1, t_1 > & - & \underset{-b}{\circ} & - & < s_3, t_3 > \end{array}$$

For more information about the table, we refer to the original paper of Brieskorn [1].

2.2. The orbifold Bogomolov-Miyaoka-Yau inequality

Let S be a normal projective surface with quotient singularities and

$$f : S' \to S$$

be a minimal resolution of S. It is well-known that quotient singularities are log-terminal singularities. Thus one can write

$$K_{S'} \underset{num}{\equiv} f^*K_S - \sum_{p \in Sing(S)} \mathcal{D}_p,$$

Table 1. Classification of finite subgroups of $GL(2, \mathbb{C})$

Type	G	$\lvert G\rvert$	$G/[G, G]$	Dual Graph $\varGamma_G$
A_{q,q_1}	C_{q,q_1}	q	$\mathbb{Z}_q$	$< q, q_1 >$ $0 < q_1 < q$, $(q, q_1) = 1$
D_{q,q_1}	$(Z_{2m}, Z_{2m}; D_q, D_q)$	$4mq$	$\mathbb{Z}_{2m} \times \mathbb{Z}_2$	$< b; 2, 1; 2, 1; q, q_1 >$ $m = (b-1)q - q_1$ odd
D_{q,q_1}	$(Z_{4m}, Z_{2m}; D_q, C_{2q})$	$4mq$	$\mathbb{Z}_{4m}$	$< b; 2, 1; 2, 1; q, q_1 >$ $m = (b-1)q - q_1$ even
T_m	$(Z_{2m}, Z_{2m}; T, T)$	$24m$	$\mathbb{Z}_{3m}$	$< b; 2, 1; 3, 2; 3, 2 >, m = 6(b-2) + 1$ $< b; 2, 1; 3, 1; 3, 1 >, m = 6(b-2) + 5$
T_m	$(Z_{2m}, Z_{2m}; T, D_2)$	$24m$	$\mathbb{Z}_{3m}$	$< b; 2, 1; 3, 1; 3, 2 >, m = 6(b-2) + 3$
O_m	$(Z_{2m}, Z_{2m}; O, O)$	$48m$	$\mathbb{Z}_{2m}$	$< b; 2, 1; 3, 2; 4, 3 >, m = 12(b-2) + 1$ $< b; 2, 1; 3, 1; 4, 3 >, m = 12(b-2) + 5$ $< b; 2, 1; 3, 2; 4, 1 >, m = 12(b-2) + 7$ $< b; 2, 1; 3, 1; 4, 1 >, m = 12(b-2) + 11$
I_m	$(Z_{2m}, Z_{2m}; I, I)$	$120m$	$\mathbb{Z}_m$	$< b; 2, 1; 3, 2; 5, 4 >, m = 30(b-2) + 1$ $< b; 2, 1; 3, 2; 5, 3 >, m = 30(b-2) + 7$ $< b; 2, 1; 3, 1; 5, 4 >, m = 30(b-2) + 11$ $< b; 2, 1; 3, 2; 5, 2 >, m = 30(b-2) + 13$ $< b; 2, 1; 3, 1; 5, 3 >, m = 30(b-2) + 17$ $< b; 2, 1; 3, 2; 5, 1 >, m = 30(b-2) + 19$ $< b; 2, 1; 3, 1; 5, 2 >, m = 30(b-2) + 23$ $< b; 2, 1; 3, 1; 5, 1 >, m = 30(b-2) + 29$

where $\mathcal{D}_p = \sum(a_j E_j)$ is an effective $\mathbb{Q}$-divisor supported on $f^{-1}(p) = \cup E_j$ with $0 \leq a_j < 1$ for each singular point p. It implies that

$$K_S^2 = K_{S'}^2 - \sum_p \mathcal{D}_p^2 = K_{S'}^2 + \sum_p \mathcal{D}_p K_{S'}.$$

Lemma 1. *If $-K_S$ is ample, then $C^2 \geq -1$ for any irreducible curve $C \subset S'$ not contracted by $f : S' \to S$.*

Proof. Note that $C(f^* K_S) < 0$ and $C(\sum \mathcal{D}_p) \geq 0$. Thus $CK_{S'} < 0$, and hence $C^2 \geq -1$. □

Also we recall the orbifold Euler characteristic

$$e_{orb}(S) := e(S) - \sum_{p \in Sing(S)} \Big(1 - \frac{1}{|G_p|}\Big),$$

where G_p is the local fundamental group of p.

The following theorem, called the orbifold Bogomolov-Miyaoka-Yau inequality, is one of the main ingredients in the proof of our main theorem.

Theorem 4 ([17], [14], [10], [13]). *Let S be a normal projective surface with quotient singularities such that K_S is nef. Then*

$$K_S^2 \leq 3e_{orb}(S).$$

In particular,

$$0 \leq e_{orb}(S).$$

The weaker inequality holds when $-K_S$ is nef.

Theorem 5 ([9]). *Let S be a normal projective surface with quotient singularities such that $-K_S$ is nef. Then*

$$0 \leq e_{orb}(S).$$

2.3. Divisors on the minimal resolution

Let S be a normal projective surface with quotient singularities and $f : S' \to S$ be a minimal resolution of S. It is well-known that the torsion-free part of the second cohomology group,

$$H^2(S', \mathbb{Z})_{free} := H^2(S', \mathbb{Z})/(torsion)$$

has a lattice structure which is unimodular. For a quotient singular point $p \in S$, let

$$R_p \subset H^2(S', \mathbb{Z})_{free}$$

be the sublattice of $H^2(S', \mathbb{Z})_{free}$ spanned by the numerical classes of the components of $f^{-1}(p)$. It is a negative definite lattice, and its discriminant group

$$\mathrm{disc}(R_p) := \mathrm{Hom}(R_p, \mathbb{Z})/R_p$$

is isomorphic to the abelianization $G_p/[G_p, G_p]$ of the local fundamental group G_p. In particular, the absolute value $|\det(R_p)|$ of the determinant of the intersection matrix of R_p is equal to the order $|G_p/[G_p, G_p]|$. Let

$$R = \oplus_{p \in Sing(S)} R_p \subset H^2(S', \mathbb{Z})_{free}$$

be the sublattice of $H^2(S',\mathbb{Z})_{free}$ spanned by the numerical classes of the exceptional curves of $f : S' \to S$. We also consider the sublattice

$$R + \langle K_{S'}\rangle \subset H^2(S',\mathbb{Z})_{free}$$

spanned by R and the canonical class $K_{S'}$. Note that

$$\mathrm{rank}(R) \le \mathrm{rank}(R + \langle K_{S'}\rangle) \le \mathrm{rank}(R) + 1.$$

Lemma 2 ([4], Lemma 3.3). *Let S be a normal projective surface with quotient singularities and $f : S' \to S$ be a minimal resolution of S. Then the following hold true.*

1. $\mathrm{rank}(R + \langle K_{S'}\rangle) = \mathrm{rank}(R)$ *if and only if K_S is numerically trivial.*
2. $\det(R + \langle K_{S'}\rangle) = \det(R)\cdot K_S^2$ *if K_S is not numerically trivial.*
3. *If in addition $b_2(S) = 1$ and K_S is not numerically trivial, then $R + \langle K_{S'}\rangle$ is a sublattice of finite index in the unimodular lattice $H^2(S',\mathbb{Z})_{free}$, in particular $|\det(R + \langle K_{S'}\rangle)|$ is a nonzero square number.*

We denote the number $|\det(R + \langle K_{S'}\rangle)|$ by D, i.e., we define

$$D := |\det(R + \langle K_{S'}\rangle)|.$$

The following will be also used in our proof.

Lemma 3. *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities such that $H_1(S^0,\mathbb{Z}) = 0$. Let $f : S' \to S$ be a minimal resolution. Then*

1. *$H^2(S',\mathbb{Z})$ is torsion free, i.e. $H^2(S',\mathbb{Z}) = H^2(S',\mathbb{Z})_{free}$,*
2. *R is a primitive sublattice of the unimodular lattice $H^2(S',\mathbb{Z})$,*
3. $\mathrm{disc}(R)$ *is a cyclic group, in particular, the orders $|G_p/[G_p,G_p]| = |\det(R_p)|$ are pairwise relatively prime,*
4. *K_S is not numerically trivial, i.e. K_S is either ample or anti-ample,*
5. *$D = |\det(R)|K_S^2$ and D is a nonzero square number,*
6. *the Picard group $Pic(S')$ is generated over $\mathbb{Z}$ by the exceptional curves and a $\mathbb{Q}$-divisor M of the form*

$$M = \frac{1}{\sqrt{D}} f^*K_S + \sum_{p\in Sing(S)} b_p e_p$$

for some integers b_p, where e_p is a generator of $\mathrm{disc}(R_p)$.

Proof. (1), (2) and (3) are easy to see (cf. [6], Proposition 2.3 and Lemma 3.4).

(4) Assume that K_S is numerically trivial. Then S' is an Enriques surface if all singularities are rational double points, and is a rational surface otherwise.

If S' is an Enriques surface, then $H_1(S^0,\mathbb{Z}) \neq 0$ since $H_1(S',\mathbb{Z}) = \mathbb{Z}/2$ (cf. Proposition 2.3 in [6]). Thus S is a rational surface, and

$$K_{S'} \underset{num}{\equiv} - \sum_{p \in Sing(S)} \mathcal{D}_p$$

with $\mathcal{D}_p \underset{num}{\not\equiv} 0$ for some p. Note that $\mathcal{D}_p$ defines an element of $R_p^* := \mathrm{Hom}(R_p,\mathbb{Z})$ and the discriminant group $\mathrm{disc}(R_p) := R_p^*/R_p$ has order $|\det(R_p)|$. Thus $|\det(R_p)|\mathcal{D}_p \in R_p$ but $\mathcal{D}_p \notin R_p$ if $\mathcal{D}_p \underset{num}{\not\equiv} 0$. Now we see that

$$\big(\prod_p |\det(R_p)|\big) K_{S'} \in R \subset H^2(S',\mathbb{Z}),$$

but $K_{S'} \notin R$. Hence the primitive closure $\bar{R}$ of R in $H^2(S',\mathbb{Z})$ is not equal to R. Now by Lemma 2.5 in [6], $H_1(S^0,\mathbb{Z}) \neq 0$.

(5) follows from (4) and Lemma 2.

(6) Note first that $Pic(S') = H^2(S',\mathbb{Z})$ and the sublattice $R \subset H^2(S',\mathbb{Z})$ generated by the exceptional curves is a primitive sublattice of corank 1. Let $R^\perp \subset H^2(S',\mathbb{Z})$ be the orthogonal complement of R. Note that $R^\perp$ is positive definite and of rank 1. Since $H^2(S',\mathbb{Z})$ is unimodular,

$$\det(R^\perp) = |\det(R)| = \prod_{p \in Sing(S)} |\det(R_p)|.$$

Note that $f^*K_S \in R^\perp$. Thus $R^\perp$ is generated by

$$v := \frac{|\det(R)|}{\sqrt{D}} f^*K_S,$$

and $\mathrm{disc}(R^\perp)$ is generated by

$$\frac{1}{\sqrt{D}} f^*K_S.$$

Also note that

$$\mathrm{disc}(R^\perp \oplus R) \cong (\mathbb{Z}/|\det(R)|) \oplus (\mathbb{Z}/|\det(R)|).$$

Thus $Pic(S')/(R^\perp \oplus R)$ is an isotropic subgroup of $\mathrm{disc}(R^\perp \oplus R)$ of order $|\det(R)|$, hence is generated by an element $M \in \mathrm{disc}(R^\perp \oplus R)$ of order $|\det(R)|$. Moreover M is the sum of a generator of $\mathrm{disc}(R^\perp)$ and a generator of $\mathrm{disc}(R)$, since $Pic(S')$ is unimodular. By replacing M by kM for a suitable choice of an integer k, we get M of the desired form

$$M = \frac{1}{\sqrt{D}} f^*K_S + \sum_{p \in Sing(S)} b_p e_p$$

for some integers b_p with $0 \leq b_p < |\det(R_p)|$, where $\sum b_p e_p$ is a generator of $\mathrm{disc}(R)$. This proves that $Pic(S')$ is generated over $\mathbb{Z}$ by R, v and M. Finally, note that

$$|\det(R)|M = v \pmod{R},$$

i.e., v is generated by M and R. Thus $Pic(S')$ is generated over $\mathbb{Z}$ by R and M. □

3. Proof of Theorem 3

Let S be a $\mathbb{Q}$-homology projective plane with 4 or more quotient singularities with $H_1(S^0, \mathbb{Z}) = 0$. By Lemma 3(3), the orders of the abelianized local fundamental groups are pairwise relatively prime. Thus by Theorem 5, one can see that S has 4 singular points and the 4-tuple of orders of the local fundamental groups must be one of the following:

1. $(2, 3, 5, q)$, $q \geq 7$,
2. $(2, 3, 7, q)$, $11 \leq q \leq 41$,
3. $(2, 3, 11, 13)$.

Table 1 shows that all non-cyclic singularities of type different from I_m have abelianized local fundamental groups of order divisible by 2 or 3.

Assume that one of the singularities is non-cyclic. By Lemma 3(3), it must be of type I_m and the other 3 singularities are cyclic of order 2, 3 and 5, respectively. Here we recall that $I_m \subset GL(2, \mathbb{C})$ is the $2m$-ary icosahedral group of order $120m$. Table 1 shows that there are 8 infinite cases of type I_m.

There are two types of order 3, $< 3, 2 >$ and $< 3, 1 >$; three types of order 5, $< 5, 4 >$, $< 5, 3 > \cong < 5, 2 >$ and $< 5, 1 >$. Thus there are exactly 48 infinite cases for possible combinations of types of singularities. That is, there are exactly 48 infinite cases for R, the sublattice of $Pic(S') = H^2(S', \mathbb{Z})$ generated by all exceptional curves, where $f : S' \to S$ is a minimal resolution. In each of the 48 cases we compute $D = |\det(R)|K_S^2$ and check if D is a square number (see Lemma 3(5)), using elementary number theoretic arguments. There remain 8 infinite cases and 2 sporadic cases, as given in Table 2 and Table 3. In both tables, the entries of the column b are the possible values of b that make D a square number.

We will explain how to compute D. First note that

$$|\det(R)| = 2 \cdot 3 \cdot 5 \cdot m = 30m.$$

To compute K_S^2, we use the equality from (2.2)

$$K_S^2 = K_{S'}^2 + \sum_p \mathcal{D}_p K_{S'}.$$

Note that S' has $H^1(S', \mathcal{O}_{S'}) = H^2(S', \mathcal{O}_{S'}) = 0$. Thus by Noether formula,

$$K_{S'}^2 = 12 - e(S') = 10 - b_2(S') = 9 - \mu,$$

where μ is the number of the exceptional curves of f.
For each singular point p, the coefficients of the $\mathbb{Q}$-divisor $\mathcal{D}_p$ can be obtained by solving the equations given by the adjunction formula

$$\mathcal{D}_p E = -K_{S'} E = 2 + E^2$$

for each exceptional curve $E \subset f^{-1}(p)$. Once we know the coefficients, we can easily compute the intersection number $\mathcal{D}_p K_{S'}$.

We first rule out the two sporadic cases.

Lemma 4. *The case* $< 2,1 > + < 3,2 > + < 5,4 > + < 8;2,1;3,2;5,3 >$ *does not occur.*

Proof. In this case, $m = 30(b-2) + 7 = 187$, so

$$|\det(R)| = 30 \cdot 187.$$

The number of exceptional curves $\mu = 13$, so $K_{S'}^2 = -4$, where $f : S' \to S$ is a minimal resolution. Let p_1, p_2, p_3, p_4 be the four singular points. Let $E_1, \dots, E_6$ be the components of $f^{-1}(p_4)$ such that

$$\begin{array}{ccccccccc} \overset{-2}{E_2} & - & \overset{-2}{E_3} & - & \overset{-8}{E_6} & - & \overset{-2}{E_5} & - & \overset{-3}{E_4} \\ & & & & | & & & & \\ & & & & \underset{-2}{E_1} & & & & \end{array}$$

is their dual graph. Solving the equations given by the adjunction formula, we get

$$K_{S'} = f^*K_S - \frac{93E_1 + 186E_6 + 62E_2 + 124E_3 + 112E_4 + 149E_5}{187}.$$

It is easy to compute that

$$K_S^2 = K_{S'}^2 + \frac{186E_6K_{S'} + 112E_4K_{S'}}{187} = -4 + \frac{186 \cdot 6 + 112}{187} = \frac{480}{187}.$$

Thus

$$D = |\det(R)| K_S^2 = 120^2.$$

Note that $K_S^2 > 3e_{orb}(S)$, so $-K_S$ is ample by the orbifold Bogomolov-Miyaoka-Yau inequality. Thus S' is a rational surface, not minimal. Also note that the divisor M from Lemma 3(6) takes the form

$$M = \frac{1}{120} f^*K_S + \sum_{p \in Sing(S)} a_p e_p.$$

Table 2.

Type of R	$D = \|det(R)\|K_S^2$	b
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,2;5,4 >$	$180(5b^2 - 50b + 79)$	none
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,2;5,3 >$	$180(5b^2 - 36b + 48)$	$b = 8$
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,1;5,4 >$	$180(5b^2 - 40b + 52)$	none
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,2;5,2 >$	$180(5b^2 - 34b + 41)$	none
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,1;5,3 >$	$180(5b^2 - 26b + 27)$	none
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,2;5,1 >$	$180(5b^2 - 20b + 18)$	none
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,1;5,2 >$	$180(5b^2 - 24b + 22)$	none
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,1;5,1 >$	$900(b-1)^2$	$b \geq 2$
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,2;5,4 >$	$36(25b^2 - 190b + 277)$	none
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,2;5,3 >$	$36(25b^2 - 120b + 134)$	none
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,1;5,4 >$	$36(25b^2 - 140b + 162)$	none
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,2;5,2 >$	$36(25b^2 - 110b + 111)$	none
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,1;5,3 >$	$36(5b-7)^2$	$b \geq 2$
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,2;5,1 >$	$36(25b^2 - 40b + 8)$	none
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,1;5,2 >$	$36(5b-6)^2$	$b \geq 2$
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,1;5,1 >$	$36(25b^2 + 10b - 37)$	none
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,2;5,4 >$	$36(25b^2 - 130b + 159)$	none
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,2;5,3 >$	$36(25b^2 - 60b + 28)$	none
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,1;5,4 >$	$36(5b-8)^2$	$b \geq 2$
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,2;5,2 >$	$36(25b^2 - 50b + 17)$	none
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,1;5,3 >$	$36(25b^2 - 10b - 37)$	none
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,2;5,1 >$	$36(25b^2 + 20b - 74)$	none
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,1;5,2 >$	$36(25b^2 - 38)$	none
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,1;5,1 >$	$36(25b^2 + 70b - 99)$	none

Let C be a (-1)-curve on S'. By Lemma 3(6), C can be written as

$$C = kM + r$$

for some integer k and some $r \in R$, hence as

$$C = \frac{k}{120} f^*K_S + C(1) + C(2) + C(3) + C(4),$$

Table 3.

Type of R	$D = \|det(R)\|K_S^2$	b
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,2;5,4 >$	$20(45b^2 - 390b + 593)$	none
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,2;5,3 >$	$20(45b^2 - 264b + 326)$	none
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,1;5,4 >$	$100(9b^2 - 60b + 74)$	none
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,2;5,2 >$	$20(45b^2 - 246b + 275)$	none
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,1;5,3 >$	$20(45b^2 - 174b + 157)$	none
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,2;5,1 >$	$100(3b - 4)^2$	$b \geq 2$
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,1;5,2 >$	$20(45b^2 - 156b + 124)$	none
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,1;5,1 >$	$20(45b^2 - 30b - 17)$	none
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,2;5,4 >$	$4(225b^2 - 1410b + 1903)$	none
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,2;5,3 >$	$4(15b - 26)^2$	$b \geq 2$
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,1;5,4 >$	$4(225b^2 - 960b + 968)$	none
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,2;5,2 >$	$4(15b - 23)^2$	$b \geq 2$
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,1;5,3 >$	$4(225b^2 - 330b + 11)$	none
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,2;5,1 >$	$4(225b^2 - 60b - 338)$	none
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,1;5,2 >$	$4(225b^2 - 240b - 46)$	none
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,1;5,1 >$	$4(225b^2 + 390b - 643)$	none
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,2;5,4 >$	$4(15b - 29)^2$	$b \geq 2$
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,2;5,3 >$	$4(225b^2 - 240b - 278)$	none
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,1;5,4 >$	$4(225b^2 - 420b + 86)$	none
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,2;5,2 >$	$4(225b^2 - 150b - 317)$	none
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,1;5,3 >$	$4(225b^2 + 210b - 763)$	none
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,2;5,1 >$	$4(225b^2 + 480b - 1076)$	$b = 2$
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,1;5,2 >$	$4(225b^2 + 300b - 712)$	none
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,1;5,1 >$	$4(225b^2 + 930b - 1201)$	none

where $C(i)$ is a $\mathbb{Q}$-divisor supported on $f^{-1}(p_i)$. Note that

$$C^2 = (\frac{k}{120}f^*K_S)^2 + C(1)^2 + C(2)^2 + C(3)^2 + C(4)^2.$$

Since $(f^*K_S)C(i) = 0$ for all i, we have

$$(f^*K_S)C = (f^*K_S)(\frac{k}{120}f^*K_S) = \frac{k}{120}K_S^2 = \frac{4k}{187}.$$

Since $-K_S$ is ample and $C \notin R$, we see that $(f^*K_S)C < 0$, hence $k < 0$. Note that $K_{S'}C = -1$. From the equality

$$K_{S'}C = (f^*K_S)C \\ -\frac{(93E_1 + 186E_6 + 62E_2 + 124E_3 + 112E_4 + 149E_5)C}{187},$$

we get

$$(93E_1 + 186E_6 + 62E_2 + 124E_3 + 112E_4 + 149E_5)C = 187 + 4k.$$

This is possible only if

$$E_6C = E_5C = E_4C = E_3C = 0, \quad E_2C = E_1C = 1, \quad k = -8.$$

Since $E_jC(4) = E_jC$ for $j = 1, ..., 6$, we obtain the coefficients of $C(4)$ by solving the equations given by the above intersection numbers.

$$C(4) = -\frac{106E_1 + 133E_2 + 79E_3 + 5E_4 + 15E_5 + 25E_6}{187} = E_1^* + E_2^*,$$

where $E_j^* \in \mathrm{Hom}(R_{p_4}, \mathbb{Z})$ is the dual vector of E_j. Thus

$$C(4)^2 = (E_1^* + E_2^*)C(4) = -\frac{106 + 133}{187}.$$

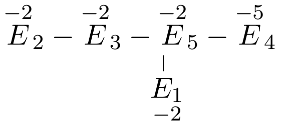
Now we have

$$\sum_{j \leq 3} C(j)^2 = C^2 - C(4)^2 - (\frac{-8f^*K_S}{120})^2 = -1 + \frac{239}{187} - \frac{32}{15 \cdot 187} > 0$$

which contradicts the negative definiteness of exceptional curves. □

Lemma 5. *The case* $< 2, 1 > + < 3, 1 > + < 5, 1 > + < 2; 2, 1; 3, 2; 5, 1 >$ *does not occur.*

Proof. The proof is similar to the previous case. In this case, $m = 19$ and $\mu = 8$, so $|\det(R)| = 30 \cdot 19$ and $K_{S'}^2 = 1$. Let B_2, B_3 be the components of $f^{-1}(p_2), f^{-1}(p_3)$. Let $E_1, \ldots, E_5$ be the components of $f^{-1}(p_4)$ such that



is their dual graph. Then

$$K_{S'} = f^*K_S - \frac{B_2}{3} - \frac{3B_3}{5} - \frac{9E_1 + 6E_2 + 12E_3 + 15E_4 + 18E_5}{19},$$

$$K_S^2 = \frac{28 \cdot 56}{15 \cdot 19}, \quad D = |\det(R)| K_S^2 = 56^2.$$

Here again by the orbifold Bogomolov-Miyaoka-Yau inequality, $-K_S$ is ample and S' is a rational surface, not minimal. Let C be a (-1)-curve on S'. Then

$$C = \frac{k}{56} f^* K_S + C(1) + C(2) + C(3) + C(4)$$

for some integer k and some $\mathbb{Q}$-divisor $C(i)$ supported on $f^{-1}(p_i)$.
Since $(f^* K_S)C = \frac{28k}{285} < 0$, we see that $k < 0$ and we get

$$95B_2C + 171B_3C + 15(9E_1 + 6E_2 + 12E_3 + 15E_4 + 18E_5)C = 285 + 28k.$$

This is impossible because $k < 0$ and $E_jC \geq 0, B_iC \geq 0$ for every i, j. □

Lemma 6. *For any of the* 8 *infinite cases,* $-K_S$ *is ample.*

Proof. For the 8 infinite cases, we compute K_S^2 as follows.

Table 4.

Type of R	K_S^2
$< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,1;5,1 >$	$\frac{30(b-1)^2}{30b-31} \geq \frac{30}{29}$
$< 2,1 > + < 3,2 > + < 5,2 > + < b;2,1;3,1;5,3 >$	$\frac{6(5b-7)^2}{5(30b-43)} \geq \frac{54}{85}$
$< 2,1 > + < 3,2 > + < 5,3 > + < b;2,1;3,1;5,2 >$	$\frac{6(5b-6)^2}{5(30b-37)} \geq \frac{96}{115}$
$< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,1;5,4 >$	$\frac{6(5b-8)^2}{5(30b-49)} \geq \frac{24}{55}$
$< 2,1 > + < 3,1 > + < 5,4 > + < b;2,1;3,2;5,1 >$	$\frac{10(3b-4)^2}{3(30b-41)} \geq \frac{40}{57}$
$< 2,1 > + < 3,1 > + < 5,2 > + < b;2,1;3,2;5,3 >$	$\frac{2(15b-26)^2}{15(30b-53)} \geq \frac{32}{105}$
$< 2,1 > + < 3,1 > + < 5,3 > + < b;2,1;3,2;5,2 >$	$\frac{2(15b-23)^2}{15(30b-47)} \geq \frac{98}{195}$
$< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,2;5,4 >$	$\frac{2(15b-29)^2}{15(30b-59)} \geq \frac{2}{15}$

In each case, $e_{orb}(S) = -1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{120m} \leq \frac{5}{120}$. From the table we see that $K_S^2 > 3e_{orb}(S)$, so $-K_S$ is ample by the orbifold Bogomolov-Miyaoka-Yau inequality. □

This completes the proof of (1) and (2) of Theorem 3. To prove the remaining parts, we need to analyze (-1)-curves on the minimal resolution S'. Note that by Lemma 1 S' contains no $(-n)$-curve with $n \geq 2$ other than the exceptional curves of $f : S' \to S$.

The following proposition will be proved case by case in the next section.

Proposition 1. *If* S *has* 4 *singularities* p_1, p_2, p_3, p_4 *of type* $< 2, 1 >$, $< 3, \alpha >$, $< 5, \beta >$, $< b; 2, 1; 3, 3 - \alpha; 5, 5 - \beta >$, $b \geq 2$, *respectively, as in Table 4, then there are three mutually disjoint* (-1)*-curves* C_1, C_2, C_3 *on* S' *such that*

1. *each C_i intersects exactly 2 components of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3) \cup f^{-1}(p_4)$ with multiplicity 1 each,*
2. *C_1 intersects the component of the branch $< 2, 1 >$ of $f^{-1}(p_4)$ and the component of $f^{-1}(p_1)$, C_2 intersects the terminal component of the branch $< 3, 3 - \alpha >$ of $f^{-1}(p_4)$ and one end component of $f^{-1}(p_2)$, and C_3 intersects the terminal component of the branch $< 5, 5 - \beta >$ of $f^{-1}(p_4)$ and one end component of $f^{-1}(p_3)$ which is a (-2)-curve if $\beta = 2$ or 4, a (-3)-curve if $\beta = 3$, and a (-5)-curve if $\beta = 1$.*

Proposition 2. *1. The surface S' can be blown down to the Hirzebruch surface F_b. Conversely, S' can be obtained by starting with F_b and blowing up inside 3 of the fibres, i.e. the blowing up starts at three centers, one on each of the 3 fibres.*

2. If two rational homology projective planes S_1 and S_2 have the same type of singularities

$$< 2, 1 > + < 3, \alpha > + < 5, \beta > + < b; 2, 1; 3, 3 - \alpha; 5, 5 - \beta >, \ b \geq 2,$$

then S_1^0 and S_2^0 are deformation equivalent.

Proof. (1) By Proposition 1 there are three mutually disjoint (-1)-curves C_1, C_2, C_3 on S' satisfying (1) and (2) of Proposition 1. By starting with them, we can blow down S' to F_b. Furthermore, the blow up process from F_b to S' is carried out inside 3 of the fibres of F_b.

(2) The blow up process from F_b to S' depends on the choice of three fibres, each with a point marked. The three marked points are the centers of the blowing up. The choice of three fibres is unique up to automorphisms of F_b, while the choice of three points, one on each of the fixed three fibres, is not unique up to automorphisms of F_b, but depends on a 2-dimensional moduli. □

This completes the proof of (3) of Theorem 3.

The following examples mentioned in Introduction were discussed in [12], Example 31.

Example 1. Consider the $2m$-ary icosahedral group

$$I_m \subset GL(2, \mathbb{C})$$

of order $120m$ in Brieskorn's list (Table 1). Let $Z \subset I_m$ be its center, then $Z \cong \mathbb{Z}_{2m}$ and $I_m/Z \cong \mathfrak{A}_5 \subset PGL(2, \mathbb{C})$, the icosahedral group. Extend the natural I_m-action on $\mathbb{C}^2$ to $\mathbb{CP}^2$. The center acts trivially on the line at infinity and $\mathbb{CP}^2/Z$ is a cone over the rational normal curve of degree $2m = |Z|$. Then

$$S_{I_m} := \mathbb{CP}^2/I_m = (\mathbb{CP}^2/Z)/\mathfrak{A}_5$$

has 4 quotient singularities, one of type $\mathbb{C}^2/I_m$ at the origin, three of order 2, 3, 5 at infinity. The fundamental group of $S_{I_m}^0$ is $\mathfrak{A}_5$. By Theorem 3 (1), the types of the 3 cyclic singularities are determined by the types of the 3 branches of the non-cyclic singularity. By Proposition 1 and 2, its minimal resolution S'_{I_m} can be blown down to the Hirzebruch surface F_b. Conversely, S'_{I_m} can be obtained by starting with F_b and blowing up inside 3 of the fibres. Here the 3 centers of the blowing up lie on a section of F_b.

In Proposition 2, the 3 centers of the blowing up lie on a section of F_b if and only if the surface S' is isomorphic to S'_{I_m} for some I_m. This completes the proof of (4) and (5) of Theorem 3.

4. Proof of Proposition 1

As before, let p_1, p_2, p_3, p_4 be the singular points of S of order 2, 3, 5, $120m$, respectively, and let $f : S' \to S$ be a minimal resolution. Let R_{p_i} be the sublattice of $H^2(S', \mathbb{Z})$ generated by all exceptional curves contained in $f^{-1}(p_i)$.

Let C be an irreducible curve on S'. By Lemma 3(6), C can be written as $C = kM + r$ for some integer k and some $r \in R$, hence as

$$C = \frac{k}{\sqrt{D}} f^* K_S + C(1) + C(2) + C(3) + C(4), \tag{1}$$

where $C(i)$ is a $\mathbb{Q}$-divisor supported on $f^{-1}(p_i)$ that is of the form

$$C(i) = a_i e_i + r_i$$

for some integer a_i and some $r_i \in R_{p_i}$, where e_i is a generator of the discriminant group disc(R_{p_i}).

Lemma 7. *Let C be an irreducible curve on S' of the form* (1).

1. $C(i)^2 = 0$ *if and only if* $C(i) = 0$ *if and only if* C *does not meet* $f^{-1}(p_i)$.
2. $C(1)^2 = -\frac{1}{2}x$ *for some integer* $x \geq 0$,
   $C(1)^2 = -\frac{1}{2}$ *if and only if* C *meets with multiplicity* 1 *the component of* $f^{-1}(p_1)$.
3. *Assume that* p_2 *is of type* $< 3, 2 >$. *Then*
   $C(2)^2 = -\frac{2}{3}y$ *for some integer* $y \geq 0$,
   $C(2)^2 = -\frac{2}{3}$ *if and only if* C *meets with multiplicity* 1 *exactly one of the two components of* $f^{-1}(p_2)$.
4. *Assume that* p_3 *is of type* $< 5, 4 >$. *Then*
   $C(3)^2 \leq -\frac{4}{5}$ *if* $C(3) \neq 0$,
   $C(3)^2 = -\frac{4}{5}$ *if and only if* C *meets with multiplicity* 1 *exactly one of the two end components of* $f^{-1}(p_3)$.

Proof. (1) The first equivalence follows from the negative definiteness of exceptional curves. Note that $EC = EC(i)$ for any curve $E \subset f^{-1}(p_i)$.
The curve C does not meet $f^{-1}(p_i)$ iff $EC = 0$ for any curve $E \subset f^{-1}(p_i)$ iff $EC(i) = 0$ for any curve $E \subset f^{-1}(p_i)$ iff $C(i) = 0$.

(2) is trivial.

(3) Let E_1, E_2 be the exceptional curves generating R_{p_2}. Take

$$e := -\frac{E_1 + 2E_2}{3} = E_2^*$$

as a generator of $\mathrm{disc}(R_{p_2})$. Then $C(2)$ is of the form $C(2) = ae + b_1E_1 + b_2E_2$ for some integers a, b_1, b_2, hence of the form $C(2) = se + tE_2$ for some integers s, t. We have

$$C(2)^2 = -\frac{2}{3}(s^2 - 3st + 3t^2).$$

It is easy to see that $y := s^2 - 3st + 3t^2 = (s - 3t/2)^2 + 3t^2/4 \geq 0$ for all s, t.

C meets exactly one of the two components of $f^{-1}(p_2)$ with multiplicity 1 iff $(E_1C(2), E_2C(2)) = (1, 0)$ or $(0, 1)$ iff $C(2) = E_1^* = 2e + E_2$ or $C(2) = E_2^* = e$ iff $(s, t) = (2, 1)$ or $(1, 0)$. Both cases satisfy $C(2)^2 = -2/3$. Conversely, if $C(2)^2 = -2/3$, then there are six solutions $(s, t) = \pm(1, 0), \pm(2, 1), \pm(1, 1)$ for the equation $y = (s - 3t/2)^2 + 3t^2/4 = 1$. Since $E_iC(2) = E_iC \geq 0$ for $i = 1, 2$, there remain only two solutions $(s, t) = (1, 0), (2, 1)$.

(4) Let E_1, E_2, E_3, E_4 be the exceptional curves generating R_{p_3}. Take

$$e := -\frac{E_1 + 2E_2 + 3E_3 + 4E_4}{5} = E_4^*$$

as a generator of $\mathrm{disc}(R_{p_3})$. Then $C(3)$ is of the form $C(3) = ae + b_1E_1 + b_2E_2 + b_3E_3 + b_4E_4$ for some integers a, b_1, b_2, b_3, b_4, hence of the form $C(3) = se + uE_2 + vE_3 + wE_4$ for some integers s, u, v, w. We have

$$\begin{aligned} C(3)^2 &= -\tfrac{4}{5}s^2 - 2u^2 - 2v^2 - 2w^2 + 2sw + 2uv + 2vw \\ &= -\tfrac{4}{5}\{(s - \tfrac{5w}{4})^2 + \tfrac{5}{2}(u - \tfrac{v}{2})^2 + \tfrac{15}{8}(v - \tfrac{2w}{3})^2 + \tfrac{5}{48}w^2\}. \end{aligned}$$

To prove the first assertion, assume that

$$(s - \frac{5w}{4})^2 + \frac{5}{2}(u - \frac{v}{2})^2 + \frac{15}{8}(v - \frac{2w}{3})^2 + \frac{5}{48}w^2 < 1.$$

We need to show that $(s, u, v, w) = (0, 0, 0, 0)$. The above inequality implies that $w^2 \leq 9$, i.e., $w = 0, \pm1, \pm2, \pm3$. If $w = 0$, then there is only one solution $(s, u, v, w) = (0, 0, 0, 0)$ to the inequality. If $w = \pm1, \pm2, \pm3$, no solution to the inequality. This proves the first assertion.

C meets exactly one of the two end components of $f^{-1}(p_3)$ with multiplicity 1 iff $(E_1C, E_2C, E_3C, E_4C) = (1, 0, 0, 0)$ or $(0, 0, 0, 1)$ iff $C(3) = E_1^* =$

$4e + E_2 + 2E_3 + 3E_4$ or $C(3) = E_4^* = e$ iff $(s,u,v,w) = (4,1,2,3)$ or $(1,0,0,0)$. Both cases satisfy $C(3)^2 = -4/5$. Conversely, if $C(3)^2 = -\frac{4}{5}$, then

$$(s - \frac{5w}{4})^2 + \frac{5}{2}(u - \frac{v}{2})^2 + \frac{15}{8}(v - \frac{2w}{3})^2 + \frac{5}{48}w^2 = 1.$$

There are ten solutions to this equation: $(s,u,v,w) = \pm(1,0,0,0)$,
$\pm(4,1,2,3)$, $\pm(1,1,1,1)$, $\pm(1,0,1,1)$, $\pm(1,0,0,1)$.
Since $E_iC(3) = E_iC \geq 0$ for $i = 1,2,3,4$, there remain only two solutions $(s,u,v,w) = (4,1,2,3), (1,0,0,0)$. $\square$

4.1. Case 1: $< 2,1 > + < 3,2 > + < 5,4 > + < b;2,1;3,1;5,1 >$*,* $b \geq 2$

In this case, the number of exceptional curves $\mu = 11$, so $K_{S'}^2 = -2$. Let $E_1, \ldots, E_4$ be the components of $f^{-1}(p_4)$ such that

$$\begin{array}{ccccc} \overset{-3}{E}_2 & - & \overset{-b}{E}_4 & - & \overset{-5}{E}_3 \\ & & | & & \\ & & \underset{-2}{E_1} & & \end{array}$$

is their dual graph. We compute

$$K_{S'} = f^*K_S - \frac{(15b-16)E_1 + (20b-21)E_2 + (24b-25)E_3 + (30b-32)E_4}{30b-31}, \tag{2}$$

$$K_S^2 = \frac{30(b-1)^2}{30b-31}, |\det(R)| = 30\cdot(30b-31), D = |\det(R)|K_S^2 = 30^2(b-1)^2.$$

We also compute the dual vectors,

$$\begin{aligned} E_1^* &= -\tfrac{1}{30b-31}\{(15b-8)E_1 + 5E_2 + 3E_3 + 15E_4\}, \\ E_2^* &= -\tfrac{1}{30b-31}\{5E_1 + (10b-7)E_2 + 2E_3 + 10E_4\}, \\ E_3^* &= -\tfrac{1}{30b-31}\{3E_1 + 2E_2 + (6b-5)E_3 + 6E_4\}. \end{aligned}$$

Claim 4.1.1. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

Case	CE_4	CE_3	CE_2	CE_1	k
(a)	0	0	0	1	-15
(b)	0	0	1	0	-10
(c)	0	1	0	0	-6

Proof. We use the same argument as in the proof of Lemma 4. First note that $(f^*K_S)C = \frac{k}{\sqrt{D}}(f^*K_S)^2 = \frac{(b-1)k}{30b-31}$. Since $-K_S$ is ample and $C \notin R$, $(f^*K_S)C < 0$, so $k < 0$. Intersecting C with (2) we get

$$\begin{aligned}&C\{(15b-16)E_1 + (20b-21)E_2 + (24b-25)E_3 + (30b-32)E_4\}\\ &= (b-1)k + 30b - 31.\end{aligned}$$

This is possible only if C satisfies one of the three cases (a), (b), (c), or the case

(d) $CE_4 = 1, CE_3 = 0, CE_2 = 0, CE_1 = 0, \quad b = 2, \quad k = -1.$

In the last case, we compute $C(4) = E_4^* = -\frac{1}{29}(15E_1 + 10E_2 + 6E_3 + 30E_4)$, so $C(4)^2 = E_4^*C(4) = -\frac{30}{29}$ and hence we get

$$\sum_{j\leq 3} C(j)^2 = C^2 - C(4)^2 - (\frac{-1}{30}f^*K_S)^2 = -1 + \frac{30}{29} - \frac{1}{30\cdot 29} > 0,$$

contradicts the negative definiteness of exceptional curves. □

Claim 4.1.2. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then C meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity is 1, and the component is

1. the component of $f^{-1}(p_1)$, if C satisfies (a),
2. one of the two components of $f^{-1}(p_2)$, if C satisfies (b),
3. one of the two end components of $f^{-1}(p_3)$, if C satisfies (c).

Proof. Assume that C satisfies (a). Then, $C(4) = E_1^*$, $C(4)^2 = E_1^*C(4) = -\frac{15b-8}{30b-31}$,
$C(1)^2 + C(2)^2 + C(3)^2 = C^2 - C(4)^2 - (\frac{-15}{30(b-1)}f^*K_S)^2 = -\frac{1}{2}$.
By Lemma 7, $C(2) = C(3) = 0$, $C(1)^2 = -\frac{1}{2}$, and C does not meet $f^{-1}(p_2) \cup f^{-1}(p_3)$, but meets the component of $f^{-1}(p_1)$ with multiplicity 1.

Assume that C satisfies (b). Then, $C(4) = E_2^*$, $C(4)^2 = E_2^*C(4) = -\frac{10b-7}{30b-31}$,
$C(1)^2 + C(2)^2 + C(3)^2 = C^2 - C(4)^2 - (\frac{-10}{30(b-1)}f^*K_S)^2 = -\frac{2}{3}$.
By Lemma 7, $C(1) = C(3) = 0$, $C(2)^2 = -\frac{2}{3}$, and C does not meet $f^{-1}(p_1) \cup f^{-1}(p_3)$, but meets one of the two components of $f^{-1}(p_2)$ with multiplicity 1.

Assume that C satisfies (c). Then, $C(4) = E_3^*$, $C(4)^2 = E_3^*C(4) = -\frac{6b-5}{30b-31}$,
$C(1)^2 + C(2)^2 + C(3)^2 = C^2 - C(4)^2 - (\frac{-6}{30(b-1)}f^*K_S)^2 = -\frac{4}{5}$.
By Lemma 7, $C(1) = C(2) = 0$, $C(3)^2 = -\frac{4}{5}$, and C does not meet $f^{-1}(p_1) \cup f^{-1}(p_2)$, but meets one of the end components of $f^{-1}(p_3)$ with multiplicity 1.
□

Claim 4.1.3. There are three, mutually disjoint, (-1)-curves C_1, C_2, C_3 satisfying (a), (b), (c) from Claim 4.1.1, respectively.

Proof. By Lemma 6, S' is a rational surface. Since $K_{S'}^2 < 8$, S' contains a (-1)-curve and can be blown down to a minimal rational surface F_n or $\mathbb{CP}^2$.

Assume that there is no (-1)-curve $C \subset S'$ meeting $f^{-1}(p_4)$. Then, since S' cannot contain a $(-l)$-curve with $l \geq 2$ other than the exceptional curves of f (Lemma 1), the configuration of $f^{-1}(p_4)$ remains the same under the blow down process to F_n or $\mathbb{CP}^2$. This is impossible, as the configuration would define a negative definite sublattice of rank 4 inside the Picard lattice of F_n or $\mathbb{CP}^2$.

Assume that there is only one (-1)-curve meeting $f^{-1}(p_4)$. Then, the 3 components of $f^{-1}(p_4)$ untouched by the (-1)-curve remain the same under the blow down process and define a negative definite sublattice of rank 3 inside the Picard lattice of F_n or $\mathbb{CP}^2$. This is impossible.

If there are only two (-1)-curve meeting $f^{-1}(p_4)$. Then the 2 components of $f^{-1}(p_4)$ untouched by the two (-1)-curves would remain the same under the blow down process and define a negative definite sublattice of rank 2 inside the Picard lattice of F_n or $\mathbb{CP}^2$. Again, this is impossible.

For the mutual disjointness, we note that

$C_1 = \frac{-15}{30(b-1)} f^*K_S + C_1(1) + E_1^*,$

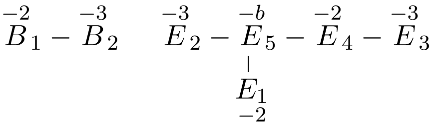
$C_2 = \frac{-10}{30(b-1)} f^*K_S + C_2(2) + E_2^*,$

$C_3 = \frac{-6}{30(b-1)} f^*K_S + C_3(3) + E_3^*.$

A direct calculation shows that $C_iC_j = 0$ for $i \neq j$. □

4.2. *Case 2:* $< 2,1 > + < 3,2 > + < 5,2 > + < b; 2,1; 3,1; 5,3 >$, $b \geq 2$

In this case, $\mu = 10$, so $K_{S'}^2 = -1$. Let B_1, B_2 be the components of $f^{-1}(p_3)$, and $E_1, \dots, E_5$ be the components of $f^{-1}(p_4)$ such that



is their dual graph. Then

$$\begin{aligned} K_{S'} = f^*K_S - \tfrac{1}{5}(B_1 + 2B_2) - \tfrac{1}{30b-43}\{&(15b-22)E_1 + (20b-29)E_2 \\ &+(18b-26)E_3 + (24b-35)E_4 + (30b-44)E_5\}, \end{aligned} \tag{3}$$

$$K_S^2 = \frac{6(5b-7)^2}{5(30b-43)}, \quad |\det(R)| = 30 \cdot (30b-43), \quad D = 6^2(5b-7)^2.$$

We also compute the dual vectors,

$B_1^* = -\frac{3B_1+B_2}{5} \qquad B_2^* = -\frac{B_1+2B_2}{5},$

$$E_1^* = -\tfrac{1}{30b-43}\{(15b-14)E_1 + 5E_2 + 3E_3 + 9E_4 + 15E_5\},$$
$$E_2^* = -\tfrac{1}{30b-43}\{5E_1 + (10b-11)E_2 + 2E_3 + 6E_4 + 10E_5\},$$
$$E_3^* = -\tfrac{1}{30b-43}\{3E_1 + 2E_2 + (12b-16)E_3 + (6b-5)E_4 + 6E_5\}.$$

Claim 4.2.1. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

Case	CE_5	CE_4	CE_3	CE_2	CE_1	CB_2	CB_1	k
(a)	0	0	0	0	1	0	0	-15
(b)	0	0	0	1	0	0	0	-10
(c)	0	0	1	0	0	0	1	-6

Proof. First note that $(f^*K_S)C = \frac{k}{\sqrt{D}}(f^*K_S)^2 = \frac{(5b-7)k}{5(30b-43)}$. Since $-K_S$ is ample and $C \notin R$, we see that $k < 0$. Intersecting C with (3) we get $(30b-43)C(B_1+2B_2) + 5C\{(15b-22)E_1 + (20b-29)E_2 + (18b-26)E_3 + (24b-35)E_4 + (30b-44)E_5\} = (5b-7)k + 5(30b-43) < 5(30b-43)$. This is possible only if C satisfies one of the three cases or the following case

(d) $CE_5 = 0, CE_4 = 1, CE_3 = CE_2 = CE_1 = 0, CB_1 = 1, CB_2 = 0,$
$b = 2, k = -1.$

In case (d), $C(3) = B_1^*$ and $C(4) = E_4^* = -\frac{1}{17}(9E_1 + 6E_2 + 7E_3 + 21E_4 + 18E_5)$, thus $C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{-1}{18}f^*K_S)^2 = -1 + \frac{3}{5} + \frac{21}{17} - \frac{1}{30\cdot 17} > 0$, contradicts the negative definiteness of exceptional curves. □

Claim 4.2.2. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then C meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity is 1, and the component is

1. the component of $f^{-1}(p_1)$, if C satisfies (a),
2. one of the two components of $f^{-1}(p_2)$, if C satisfies (b),
3. the component B_1 of $f^{-1}(p_3)$, if C satisfies (c).

Proof. Assume that C satisfies (a). Then, $C(3) = 0$ and $C(4) = E_1^*$, so

$$C(1)^2 + C(2)^2 = C^2 - C(4)^2 - (\tfrac{-15}{6(5b-7)}f^*K_S)^2 = -\tfrac{1}{2}.$$

By Lemma 7, $C(2) = 0$ and $C(1)^2 = -\frac{1}{2}$.

Assume that C satisfies (b). Then, $C(3) = 0$ and $C(4) = E_2^*$, so

$$C(1)^2 + C(2)^2 = C^2 - C(4)^2 - (\tfrac{-10}{6(5b-7)}f^*K_S)^2 = -\tfrac{2}{3}.$$

By Lemma 7, $C(1) = 0$ and $C(2)^2 = -\frac{2}{3}$.

Assume that C satisfies (c). Then, $C(3) = B_1^* = -\frac{3B_1+B_2}{5}$ and $C(4) = E_3^*$, so

$$C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\tfrac{-6}{6(5b-7)}f^*K_S)^2 = 0.$$

By the negative definiteness, $C(1) = C(2) = 0$. □

By the same proof as in the previous case, we see that there are three, mutually disjoint, (-1)-curves C_1, C_2, C_3 satisfying (a), (b), (c) from Claim 4.2.1, respectively.

4.3. Case 3: $< 2,1 > + < 3,2 > + < 5,3 > + < b; 2,1; 3,1; 5,2 >$, $b \geq 2$

In this case, $\mu = 10$, so $K_{S'}^2 = -1$. Let B_1, B_2 be the components of $f^{-1}(p_3)$, and $E_1, \ldots, E_5$ be the components of $f^{-1}(p_4)$ such that

$$\begin{array}{ccccccccccccc} \overset{-2}{B_1} & - & \overset{-3}{B_2} & \quad & \overset{-3}{E_2} & - & \overset{-b}{E_5} & - & \overset{-3}{E_4} & - & \overset{-2}{E_3} \\ & & & & & & | & & & & \\ & & & & & & \underset{-2}{E_1} & & & & \end{array}$$

is their dual graph. Then

$$\begin{aligned} K_{S'} = f^*K_S - \tfrac{1}{5}(B_1+2B_2) - \tfrac{1}{30b-37}\{&(15b-19)E_1 + (20b-25)E_2 \\ &+ (12b-15)E_3 + (24b-30)E_4 + (30b-38)E_5\}, \end{aligned} \tag{4}$$

$$K_S^2 = \frac{6(5b-6)^2}{5(30b-37)}, \quad |\det(R)| = 30 \cdot (30b-37), \quad D = 6^2(5b-6)^2.$$

We also compute the dual vectors,

$$\begin{aligned} B_1^* &= -\tfrac{3B_1+B_2}{5} \qquad B_2^* = -\tfrac{B_1+2B_2}{5}, \\ E_1^* &= -\tfrac{1}{30b-37}\{(15b-11)E_1 + 5E_2 + 3E_3 + 6E_4 + 15E_5\}, \\ E_2^* &= -\tfrac{1}{30b-37}\{5E_1 + (10b-9)E_2 + 2E_3 + 4E_4 + 10E_5\}, \\ E_3^* &= -\tfrac{1}{30b-37}\{3E_1 + 2E_2 + (18b-21)E_3 + (6b-5)E_4 + 6E_5\}. \end{aligned}$$

Claim 4.3.1. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

Case	CE_5	CE_4	CE_3	CE_2	CE_1	CB_2	CB_1	k
(a)	0	0	0	0	1	0	0	-15
(b)	0	0	0	1	0	0	0	-10
(c)	0	0	1	0	0	1	0	-6

Proof. Since $(f^*K_S)C = \frac{(5b-6)k}{5(30b-37)} < 0$, $k < 0$. Intersecting C with (4) we get $(30b-37)C(B_1+2B_2) + 5C\{(15b-19)E_1 + (20b-25)E_2 + (12b-15)E_3 + (24b-30)E_4 + (30b-38)E_5\} = (5b-6)k + 5(30b-37) < 5(30b-37)$. This is possible only if C satisfies one of the three cases or the following case

(d) $CE_5 = 0, CE_4 = 0, CE_3 = 1, CE_2 = 0, CE_1 = 0, CB_1 = 2,$
$CB_2 = 0, k = -6.$

In the last case, $C(3) = 2B_1^*$ and $C(4) = E_3^*$, so $C(3)^2 = -\frac{12}{5}$ and $C(4)^2 = -\frac{18b-21}{30b-37}$, hence $C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{-6}{6(5b-6)} f^*K_S)^2 > 0$, which contradicts the negative definiteness of exceptional curves. □

Claim 4.3.2. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then C meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity is 1, and the component is

1. the component of $f^{-1}(p_1)$, if C satisfies (a),
2. one of the two components of $f^{-1}(p_2)$, if C satisfies (b),
3. the component B_2 of $f^{-1}(p_3)$, if C satisfies (c).

Proof. Assume that C satisfies (a). Then, $C(3) = 0$ and $C(4) = E_1^*$, so $C(1)^2 + C(2)^2 = -1 + \frac{15b-11}{30b-37} - (\frac{-15}{6(5b-6)} f^*K_S)^2 = -\frac{1}{2}$. By Lemma 7, $C(2) = 0$ and $C(1)^2 = -\frac{1}{2}$.

Assume that C satisfies (b). Then, $C(3) = 0$ and $C(4) = E_2^*$, so $C(1)^2 + C(2)^2 = -1 + \frac{10b-9}{30b-37} - (\frac{-10}{6(5b-6)} f^*K_S)^2 = -\frac{2}{3}$. By Lemma 7, $C(1) = 0$ and $C(2)^2 = -\frac{2}{3}$.

Assume that C satisfies (c). Then, $C(3) = B_2^*$ and $C(4) = E_3^*$, so $C(1)^2 + C(2)^2 = -1 + \frac{2}{5} + \frac{18b-21}{30b-37} - (\frac{-6}{6(5b-6)} f^*K_S)^2 = 0$. By the negative definiteness, $C(1) = C(2) = 0$. □

The same proof as in the previous cases shows that there are three, mutually disjoint, (-1)-curves C_1, C_2, C_3 satisfying (a), (b), (c) from Claim 4.3.1, respectively.

4.4. *Case 4:* $< 2,1 > + < 3,2 > + < 5,1 > + < b;2,1;3,1;5,4 >$, $b \geq 2$

In this case, $\mu = 11$, so $K_{S'}^2 = -2$. Let B be the component of $f^{-1}(p_3)$, and $E_1, \ldots, E_7$ be the components of $f^{-1}(p_4)$ such that

$$\begin{array}{ccccccccccc} \overset{-3}{E_2} & - & \overset{-b}{E_7} & - & \overset{-2}{E_6} & - & \overset{-2}{E_5} & - & \overset{-2}{E_4} & - & \overset{-2}{E_3} \\ & & | & & & & & & & & \\ & & \underset{-2}{E_1} & & & & & & & & \end{array}$$

is their dual graph. Then

$$\begin{aligned} K_{S'} = f^*K_S - \tfrac{3}{5}B - \tfrac{1}{30b-49}\{&(15b-25)E_1 + (20b-33)E_2 \\ &+(6b-10)E_3 + (12b-20)E_4 + (18b-30)E_5 \\ &+(24b-40)E_6 + (30b-50)E_7\}, \end{aligned} \tag{5}$$

$$K_S^2 = \frac{6(5b-8)^2}{5(30b-49)}, \quad |\det(R)| = 30 \cdot (30b-49), \quad D = 6^2(5b-8)^2.$$

We also compute the dual vectors,

$$\begin{aligned} E_1^* &= -\tfrac{1}{30b-49}\{(15b-17)E_1 + 5E_2 + 3E_3 + 6E_4 + 9E_5 + 12E_6 + 15E_7\}, \\ E_2^* &= -\tfrac{1}{30b-49}\{5E_1 + (10b-13)E_2 + 2E_3 + 4E_4 + 6E_5 + 8E_6 + 10E_7\}, \end{aligned}$$

$$E_3^* = -\frac{1}{30b-49}\{\,3E_1 + 2E_2 + (24b-38)E_3 + (18b-27)E_4 \\ +(12b-16)E_5 + (6b-5)E_6 + 6E_7\}.$$

Claim 4.4.1. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

Case	CE_7	CE_6	CE_5	CE_4	CE_3	CE_2	CE_1	CB	k
(a)	0	0	0	0	0	0	1	0	−15
(b)	0	0	0	0	0	1	0	0	−10
(c)	0	0	0	0	1	0	0	1	−6

Proof. Since $(f^*K_S)C = \frac{(5b-8)k}{5(30b-49)} < 0$, $k < 0$. Intersecting C with (5) we get $3(30b-49)CB + 5C\{(15b-25)E_1 + (20b-33)E_2 + (6b-10)E_3 + (12b-20)E_4 + (18b-30)E_5 + (24b-40)E_6 + (30b-50)E_7\} = (5b-8)k + 5(30b-49) < 5(30b-49)$.

This is possible only if C satisfies one of the three cases, or one of the two cases:

Case	CE_7	CE_6	CE_5	CE_4	CE_3	CE_2	CE_1	CB	k	b
(d)	0	0	0	0	2	0	0	1	−1	2
(e)	0	0	0	1	0	0	0	1	−1	2

In Case (d), $C(3) = B^* = -\frac{1}{5}B$ and $C(4) = 2E_3^*$, thus
$C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{-1}{12}f^*K_S)^2 = -1 + \frac{1}{5} + \frac{40}{11} - \frac{1}{30\cdot 11} > 0$.
In Case (e), $C(3) = -\frac{1}{5}B$ and $C(4) = E_4^* = -\frac{1}{11}(6E_1 + 4E_2 + 9E_3 + 18E_4 + 16E_5 + 14E_6 + 12E_7)$, thus
$C(1)^2 + C(2)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{-1}{12}f^*K_S)^2 = -1 + \frac{1}{5} + \frac{18}{11} - \frac{1}{30\cdot 11} > 0$.
Both contradict the negative definiteness of exceptional curves. □

Claim 4.4.2. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then C meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

1. the component of $f^{-1}(p_1)$, if C satisfies (a),
2. one of the two components of $f^{-1}(p_2)$, if C satisfies (b),
3. the component B of $f^{-1}(p_3)$, if C satisfies (c).

Proof. Assume that C satisfies (a). Then, $C(3) = 0$ and $C(4) = E_1^*$, so $C(4)^2 = -\frac{15b-17}{30b-49}$ and $C(1)^2 + C(2)^2 = C^2 - C(4)^2 - (\frac{-15}{6(5b-8)}f^*K_S)^2 = -\frac{1}{2}$. By Lemma 7, $C(2) = 0$ and $C(1)^2 = -\frac{1}{2}$.

Assume that C satisfies (b). Then, $C(3) = 0$ and $C(4) = E_2^*$, so $C(1)^2 + C(2)^2 = -1 + \frac{10b-13}{30b-49} - (\frac{-10}{6(5b-8)}f^*K_S)^2 = -\frac{2}{3}$. By Lemma 7, $C(1) = 0$ and $C(2)^2 = -\frac{2}{3}$.

Assume that C satisfies (c). Then, $C(3) = -\frac{1}{5}B$ and $C(4) = E_3^*$, so $C(1)^2 + C(2)^2 = -1 + \frac{1}{5} + \frac{24b-38}{30b-49} - (\frac{-6}{6(5b-8)}f^*K_S)^2 = 0$. By the negative definiteness, $C(1) = C(2) = 0$. □

Similarly, we see that there are three, mutually disjoint, (-1)-curves C_1, C_2, C_3 satisfying (a), (b), (c) from Claim 4.4.1, respectively.

4.5. *Case 5:* $< 2, 1 > + < 3, 1 > + < 5, 4 > + < b; 2, 1; 3, 2; 5, 1 >$, $b \geq 2$

In this case, $\mu = 11$, so $K_{S'}^2 = -2$. Let B be the component of $f^{-1}(p_2)$, and $E_1, \ldots, E_5$ be the components of $f^{-1}(p_4)$ such that

$$\begin{array}{ccccccc} \overset{-2}{E}_2 & - & \overset{-2}{E}_3 & - & \overset{-b}{E}_5 & - & \overset{-5}{E}_4 \\ & & & & | & & \\ & & & & \underset{-2}{E_1} & & \end{array}$$

is their dual graph. Then

$$\begin{array}{l} K_{S'} = f^*K_S - \frac{1}{3}B - \frac{1}{30b-41}\{(15b-21)E_1 + (10b-14)E_2 \\ \qquad +(20b-28)E_3 + (24b-33)E_4 + (30b-42)E_5\}, \end{array} \tag{6}$$

$$K_S^2 = \frac{10(3b-4)^2}{3(30b-41)}, \quad |\det(R)| = 30 \cdot (30b-41), \quad D = 10^2(3b-4)^2.$$

We also compute the dual vectors,

$$\begin{array}{l} E_1^* = -\frac{1}{30b-41}\{(15b-13)E_1 + 5E_2 + 10E_3 + 3E_4 + 15E_5\}, \\ E_2^* = -\frac{1}{30b-41}\{5E_1 + (20b-24)E_2 + (10b-7)E_3 + 2E_4 + 10E_5\}, \\ E_4^* = -\frac{1}{30b-41}\{3E_1 + 2E_2 + 4E_3 + (6b-7)E_4 + 6E_5\}. \end{array}$$

Claim 4.5.1. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

Case	CE_5	CE_4	CE_3	CE_2	CE_1	CB	k
(a)	0	0	0	0	1	0	-15
(b)	0	0	0	1	0	1	-10
(c)	0	1	0	0	0	0	-6

Proof. Since $(f^*K_S)C = \frac{(3b-4)k}{3(30b-41)} < 0$, $k < 0$. Intersecting C with (6) we get $(30b-41)CB + 3C\{(15b-21)E_1 + (10b-14)E_2 + (20b-28)E_3 + (24b-33)E_4 + (30b-42)E_5\} = (3b-4)k + 3(30b-41) < 3(30b-41)$.
This is possible only if C satisfies one of the three cases, or one of the following three cases:

Case	CE_6	CE_5	CE_4	CE_3	CE_2	CE_1	CB	k	b
(d)	0	0	0	1	0	0	1	-1	2
(e)	0	0	0	0	2	0	1	-1	2
(f)	0	0	0	0	1	1	0	-6	2

In Case (d), $C(2) = -\frac{1}{3}B$ and $C(4) = E_3^* = -\frac{1}{19}(10E_1 + 13E_2 + 26E_3 + 4E_4 + 20E_5)$, thus $C(1)^2 + C(3)^2 = C^2 - C(2)^2 - C(4)^2 - (\frac{-1}{20}f^*K_S)^2 = -1 + \frac{1}{3} + \frac{26}{19} - \frac{1}{30\cdot 19} > 0$.
In Case (e), $C(2) = -\frac{1}{3}B$ and $C(4) = 2E_2^*$, thus
$C(1)^2 + C(3)^2 = -1 + \frac{1}{3} + \frac{64}{19} - \frac{1}{30\cdot 19} > 0$.
In Case (f), $C(2) = 0$ and $C(4) = E_1^* + E_2^*$, thus $C(1)^2 + C(3)^2 = -1 + \frac{43}{19} - \frac{36}{30\cdot 19} > 0$. All these cases lead to a contradiction. □

Claim 4.5.2. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then C meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

1. the component of $f^{-1}(p_1)$, if C satisfies (a),
2. the component of B of $f^{-1}(p_2)$, if C satisfies (b),
3. one of the two end components of $f^{-1}(p_3)$, if C satisfies (c).

Proof. Assume that C satisfies (a). Then, $C(2) = 0$ and $C(4) = E_1^*$, so $C(4)^2 = -\frac{15b-13}{30b-41}$, hence $C(1)^2 + C(3)^2 = C^2 - C(4)^2 - (\frac{-15}{10(3b-4)}f^*K_S)^2 = -\frac{1}{2}$. By Lemma 7, $C(3) = 0$ and $C(1)^2 = -\frac{1}{2}$.

Assume that C satisfies (b). Then, $C(2) = -\frac{1}{3}B$ and $C(4) = E_2^*$, so $C(1)^2 + C(3)^2 = -1 + \frac{1}{3} + \frac{20b-24}{30b-41} - (\frac{-10}{10(3b-4)}f^*K_S)^2 = 0$. By the negative definiteness, $C(1) = C(3) = 0$.

Assume that C satisfies (c). Then, $C(2) = 0$ and $C(4) = E_4^*$, so $C(1)^2 + C(3)^2 = -1 + \frac{6b-7}{30b-41} - (\frac{-6}{10(3b-4)}f^*K_S)^2 = -\frac{4}{5}$. By Lemma 7, $C(1) = 0$ and $C(3)^2 = -\frac{4}{5}$. □

Similarly, we see that there are three, mutually disjoint, (-1)-curves C_1, C_2, C_3 satisfying (a), (b), (c) from Claim 4.5.1, respectively. In this case,
$C_1 = \frac{-15}{10(3b-4)}f^*K_S + C_1(1) + E_1^*$, $C_2 = \frac{-10}{10(3b-4)}f^*K_S + C_2(2) + E_2^*$,
$C_3 = \frac{-6}{10(3b-4)}f^*K_S + C_3(3) + E_4^*$.

4.6. Case 6: $<2,1> + <3,1> + <5,2> + <b;2,1;3,2;5,3>$, $b \geq 2$

In this case, $\mu = 10$, so $K_{S'}^2 = -1$. Let B be the component of $f^{-1}(p_2)$, B_2, B_3 be the components of $f^{-1}(p_3)$, and $E_1, \ldots, E_6$ be the components of $f^{-1}(p_4)$ such that

$$\begin{array}{ccccccccccccc} \overset{-2}{B_2} & - & \overset{-3}{B_3} & \quad & \overset{-2}{E_2} & - & \overset{-2}{E_3} & - & \overset{-b}{E_6} & - & \overset{-2}{E_5} & - & \overset{-3}{E_4} \\ &&&&&&&& | &&&& \\ &&&&&&&& \underset{-2}{E_1} &&&& \end{array}$$

is their dual graph. Then

$$\begin{aligned} K_{S'} = f^*K_S - \tfrac{1}{3}B - \tfrac{1}{5}(B_2 + 2B_3) - \tfrac{1}{30b-53}\{&(15b-27)E_1 \\ &+(10b-18)E_2 + (20b-36)E_3 + (18b-32)E_4 \\ &+(24b-43)E_5 + (30b-54)E_6\}, \end{aligned} \tag{7}$$

$$K_S^2 = \frac{2(15b-26)^2}{15(30b-53)}, \quad |\det(R)| = 30 \cdot (30b-53), \quad D = 2^2(15b-26)^2.$$

We also compute the dual vectors,

$$\begin{aligned} B_2^* &= -\tfrac{3B_2+B_3}{5} \qquad B_3^* = -\tfrac{B_2+2B_3}{5}, \\ E_1^* &= -\tfrac{1}{30b-53}\{(15b-19)E_1 + 5E_2 + 10E_3 + 3E_4 + 9E_5 + 15E_6\}, \\ E_2^* &= -\tfrac{1}{30b-53}\{5E_1 + (20b-32)E_2 + (10b-11)E_3 + 2E_4 + 6E_5 + 10E_6\}, \\ E_4^* &= -\tfrac{1}{30b-53}\{3E_1 + 2E_2 + 4E_3 + (12b-20)E_4 + (6b-7)E_5 + 6E_6\}. \end{aligned}$$

Claim 4.6.1. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

Case	CE_6	CE_5	CE_4	CE_3	CE_2	CE_1	CB_3	CB_2	CB	k
(a)	0	0	0	0	0	1	0	0	0	-15
(b)	0	0	0	0	1	0	0	0	1	-10
(c)	0	0	1	0	0	0	0	1	0	-6

Proof. Since $(f^*K_S)C = \frac{(15b-26)k}{15(30b-53)} < 0$, $k < 0$. Intersecting C with (7) we get
$(30b-53)C(5B+3B_2+6B_3)+15C\{(15b-27)E_1+(10b-18)E_2+(20b-36)E_3$
$+(18b-32)E_4+(24b-43)E_5+(30b-54)E_6\} = (15b-26)k+15(30b-53)$.
This is possible only if C satisfies one of the three cases, or one of the following five cases:

Case	CE_6	CE_5	CE_4	CE_3	CE_2	CE_1	CB_3	CB_2	CB	k	b
(d)	0	0	0	0	1	0	0	3	0	-3	2
(e)	0	0	0	0	1	0	1	1	0	-3	2
(f)	0	0	0	0	0	1	0	1	1	-1	2
(g)	0	0	0	0	2	0	0	1	0	-6	2
(h)	0	0	0	1	0	0	0	1	0	-6	2

In Case (d), $C(2) = 0, C(3) = 3B_2^*$ and $C(4) = E_2^*$, thus
$C(1)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{-3}{8}f^*K_S)^2 = -1 + \frac{27}{5} + \frac{8}{7} - \frac{9}{30\cdot 7} > 0$.
In Case (e), $C(2) = 0, C(3) = B_2^* + B_3^* = -\frac{4B_2+3B_3}{5}$ and $C(4) = E_2^*$, thus
$C(1)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{-3}{8}f^*K_S)^2 = -1 + \frac{7}{5} + \frac{8}{7} - \frac{9}{30\cdot 7} > 0$.
In Case (f), $C(2) = -\frac{1}{3}B, C(3) = B_2^*$ and $C(4) = E_1^*$, thus

$C(1)^2 = C^2 - C(2)^2 - C(3)^2 - C(4)^2 - (\frac{-1}{8}f^*K_S)^2$
$\quad = -1 + \frac{1}{3} + \frac{3}{5} + \frac{11}{7} - \frac{1}{30\cdot 7} > 0.$
In Case (g), $C(2) = 0, C(3) = B_2^*$ and $C(4) = 2E_2^*$, thus
$C(1)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{-6}{8}f^*K_S)^2 = -1 + \frac{3}{5} + \frac{32}{7} - \frac{36}{30\cdot 7} > 0.$
In Case (h), $C(2) = 0, C(3) = B_2^*$ and $C(4) = E_3^* = -\frac{1}{7}(10E_1 + 9E_2 + 18E_3 + 4E_4 + 12E_5 + 20E_6)$, thus
$C(1)^2 = C^2 - C(3)^2 - C(4)^2 - (\frac{-6}{8}f^*K_S)^2 = -1 + \frac{3}{5} + \frac{18}{7} - \frac{36}{30\cdot 7} > 0.$
All contradict the negative definiteness of exceptional curves. □

Claim 4.6.2. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then C meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

1. the component of $f^{-1}(p_1)$, if C satisfies (a),
2. the component B of $f^{-1}(p_2)$, if C satisfies (b),
3. the component B_2 of $f^{-1}(p_3)$, if C satisfies (c).

Proof. Assume that C satisfies (a). Then, $C(2) = C(3) = 0$ and $C(4) = E_1^*$, so $C(4)^2 = -\frac{15b-19}{30b-53}$, hence $C(1)^2 = C^2 - C(4)^2 - (\frac{-15}{2(15b-26)}f^*K_S)^2 = -\frac{1}{2}$.

Assume that C satisfies (b). Then, $C(2) = -\frac{1}{3}B, C(3) = 0$ and $C(4) = E_2^*$, so $C(1)^2 = -1 + \frac{1}{3} + \frac{20b-32}{30b-53} - (\frac{-10}{2(15b-26)}f^*K_S)^2 = 0$. Hence $C(1) = 0$.

Assume that C satisfies (c). In this case, $C(2) = 0$, $C(3) = B_2^*$ and $C(4) = E_4^*$, so $C(1)^2 = -1 + \frac{3}{5} + \frac{12b-20}{30b-53} - (\frac{-6}{2(15b-26)}f^*K_S)^2 = 0$. Hence $C(1) = 0$. □

Similarly, we see that there are three, mutually disjoint, (-1)-curves C_1, C_2, C_3 satisfying (a), (b), (c) from Claim 4.6.1, respectively.

4.7. Case 7: $<2,1> + <3,1> + <5,3> + <b;2,1;3,2;5,2>$, $b \geq 2$

In this case, $\mu = 10$, so $K_{S'}^2 = -1$. Let B be the component of $f^{-1}(p_2)$, B_2, B_3 be the components of $f^{-1}(p_3)$, and $E_1, \ldots, E_6$ be the components of $f^{-1}(p_4)$ such that

$$\begin{array}{ccccccccccccc}
\overset{-2}{B_2} & - & \overset{-3}{B_3} & \quad & \overset{-2}{E_2} & - & \overset{-2}{E_3} & - & \overset{-b}{E_6} & - & \overset{-3}{E_5} & - & \overset{-2}{E_4} \\
 & & & & & & & & | & & & & \\
 & & & & & & & & \underset{-2}{E_1} & & & &
\end{array}$$

is their dual graph. Then

$$\begin{aligned}
K_{S'} = f^*K_S - \tfrac{1}{3}B - \tfrac{1}{5}(B_2 + 2B_3) - \tfrac{1}{30b-47}\{&(15b-24)E_1 \\
+(10b-16)E_2 + (20b-32)E_3 + &(12b-19)E_4 \\
+(24b-38)E_5 + (30b-48)E_6\},&
\end{aligned} \tag{8}$$

$$K_S^2 = \frac{2(15b-23)^2}{15(30b-47)}, \quad |\det(R)| = 30\cdot(30b-47), \quad D = 2^2(15b-23)^2.$$

We also compute the dual vectors,

$$\begin{aligned}
B_2^* &= -\tfrac{3B_2+B_3}{5} \qquad B_3^* = -\tfrac{B_2+2B_3}{5},\\
E_1^* &= -\tfrac{1}{30b-47}\{(15b-16)E_1+5E_2+10E_3+3E_4+6E_5+15E_6\},\\
E_2^* &= -\tfrac{1}{30b-47}\{5E_1+(20b-28)E_2+(10b-9)E_3+2E_4+4E_5+10E_6\},\\
E_4^* &= -\tfrac{1}{30b-47}\{3E_1+2E_2+4E_3+(18b-27)E_4+(6b-7)E_5+6E_6\}.
\end{aligned}$$

Claim 4.7.1. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

Case	CE_6	CE_5	CE_4	CE_3	CE_2	CE_1	CB_3	CB_2	CB	k
(a)	0	0	0	0	0	1	0	0	0	-15
(b)	0	0	0	0	1	0	0	0	1	-10
(c)	0	0	1	0	0	0	1	0	0	-6

Proof. Since $(f^*K_S)C = \frac{(15b-23)k}{15(30b-47)} < 0$, $k < 0$. Intersecting C with (8) we get
$(30b-47)C(5B+3B_2+6B_3)+15C\{(15b-24)E_1+(10b-16)E_2+(20b-32)E_3+(12b-19)E_4+(24b-38)E_5+(30b-48)E_6\} = (15b-23)k+15(30b-47)$.
This is possible only if C satisfies one of the three cases, or the case
(d) $CE_6 = CE_5 = 0, CE_4 = 1, CE_3 = 0, CE_2 = 1, CE_1 = 0, CB_3 = 0, CB_2 = 1, CB = 0, \quad b = 2, \quad k = -3$.
In the last case, $C(2) = 0, C(3) = B_2^*$ and $C(4) = E_2^* + E_4^*$, thus

$$C(1)^2 = C^2 - C(3)^2 - C(4)^2 - (\tfrac{-3}{14}f^*K_S)^2 = -1 + \tfrac{3}{5} + \tfrac{25}{13} - \tfrac{9}{30\cdot 13} > 0,$$

which contradicts the negative definiteness of exceptional curves. □

Claim 4.7.2. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then C meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

1. the component of $f^{-1}(p_1)$, if C satisfies (a),
2. the component B of $f^{-1}(p_2)$, if C satisfies (b),
3. the component B_3 of $f^{-1}(p_3)$, if C satisfies (c).

Proof. Assume that C satisfies (a). Then, $C(2) = C(3) = 0$ and $C(4) = E_1^*$, so $C(4)^2 = -\frac{15b-16}{30b-53}$, hence $C(1)^2 = C^2 - C(4)^2 - (\frac{-15}{2(15b-23)}f^*K_S)^2 = -\frac{1}{2}$.

Assume that C satisfies (b). Then, $C(2) = -\frac{1}{3}B, C(3) = 0$ and $C(4) = E_2^*$, so $C(1)^2 = -1 + \frac{1}{3} + \frac{20b-28}{30b-47} - (\frac{-10}{2(15b-23)}f^*K_S)^2 = 0$. Hence $C(1) = 0$.

Assume that C satisfies (c). In this case, $C(2) = 0$, $C(3) = B_3^*$ and $C(4) = E_4^*$, so $C(1)^2 = -1 + \frac{2}{5} + \frac{18b-27}{30b-47} - (\frac{-6}{2(15b-23)}f^*K_S)^2 = 0$. Hence $C(1) = 0$.
□

Similarly, we see that there are three, mutually disjoint, (-1)-curves C_1, C_2, C_3 satisfying (a), (b), (c) from Claim 4.7.1, respectively.

4.8. Case 8: $< 2,1 > + < 3,1 > + < 5,1 > + < b;2,1;3,2;5,4 >$, $b \geq 2$

In this case, $\mu = 11$, so $K_{S'}^2 = -2$. Let B, B_2 be the components of $f^{-1}(p_2)$, $f^{-1}(p_3)$, and $E_1, \dots, E_8$ be the components of $f^{-1}(p_4)$ such that

$$\begin{array}{ccccccccccccc} \overset{-2}{E_2} & - & \overset{-2}{E_3} & - & \overset{-b}{E_8} & - & \overset{-2}{E_7} & - & \overset{-2}{E_6} & - & \overset{-2}{E_5} & - & \overset{-2}{E_4} \\ & & & & \vert & & & & & & & & \\ & & & & \underset{-2}{E_1} & & & & & & & & \end{array}$$

is their dual graph. Then

$$\begin{aligned} K_{S'} = f^*K_S - \tfrac{1}{3}B - \tfrac{3}{5}B_2 - \tfrac{b-2}{30b-59}(&15E_1 + 10E_2 + 20E_3 + 6E_4 \\ &+12E_5 + 18E_6 + 24E_7 + 30E_8), \end{aligned} \tag{9}$$

$$K_S^2 = \frac{2(15b-29)^2}{15(30b-59)}, \quad |\det(R)| = 30 \cdot (30b-59), \quad D = 2^2(15b-29)^2.$$

We also compute the dual vectors,

$$E_1^* = -\tfrac{1}{30b-59}\{(15b-22)E_1+5E_2+10E_3+3E_4+6E_5+9E_6+12E_7+15E_8\},$$

$$\begin{aligned} E_2^* = -\tfrac{1}{30b-59}\{\, &5E_1 + (20b-36)E_2 + (10b-13)E_3 + 2E_4 + 4E_5 + 6E_6 \\ &+8E_7 + 10E_8\}, \end{aligned}$$

$$\begin{aligned} E_4^* = -\tfrac{1}{30b-59}\{\, &3E_1 + 2E_2 + 4E_3 + (24b-46)E_4 + (18b-33)E_5 \\ &+(12b-20)E_6 + (6b-7)E_7 + 6E_8\}. \end{aligned}$$

Claim 4.8.1. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then it satisfies one of the following three cases:

Case	CE_8	CE_7	CE_6	CE_5	CE_4	CE_3	CE_2	CE_1	CB_2	CB	k
(a)	0	0	0	0	0	0	0	1	0	0	-15
(b)	0	0	0	0	0	0	1	0	0	1	-10
(c)	0	0	0	0	1	0	0	0	1	0	-6

Proof. Since $(f^*K_S)C = \frac{(15b-29)k}{15(30b-59)} < 0$, $k < 0$. Intersecting C with (9) we get
$(30b-59)C(5B+9B_2) + 15(b-2)C\{15E_1 + 10E_2 + 20E_3 + 6E_4 + 12E_5 + 18E_6 + 24E_7 + 30E_8\} = (15b-29)k + 15(30b-59) < 15(30b-59)$.
This is possible only if C satisfies one of the three cases, or the case

$$\text{(d)}\ \ CB_2 = CB = 1, b = 2, k = -1, \ \ (CE_i \text{ are not determined}).$$

In case (d), $C(2) = -\frac{1}{3}B$ and $C(3) = -\frac{1}{5}B_2$, thus

$$C(1)^2 + C(4)^2 = C^2 - C(2)^2 - C(3)^2 - (\tfrac{-1}{2} f^* K_S)^2 = -\tfrac{1}{2}.$$

Also note that in this case the sublattice $R_{p_4} \subset H^2(S', \mathbb{Z})$ generated by the components of $f^{-1}(p_4)$ is a negative definite unimodular lattice of rank 8. In particular, $R^*_{p_4} = R_{p_4}$, so $C(4) \in R_{p_4}$ and $C(4)^2$ is a non-positive even integer. By Lemma 7, $C(4)^2 = 0$. Thus C does not meet $f^{-1}(p_4)$, contradicts the assumption. □

Claim 4.8.2. Let C be a (-1)-curve of the form (1). Suppose that C meets $f^{-1}(p_4)$. Then C meets only one component of $f^{-1}(p_1) \cup f^{-1}(p_2) \cup f^{-1}(p_3)$, the intersection multiplicity with the component is 1, and the component is

1. the component of $f^{-1}(p_1)$, if C satisfies (a),
2. the component B of $f^{-1}(p_2)$, if C satisfies (b),
3. the component B_2 of $f^{-1}(p_3)$, if C satisfies (c).

Proof. Assume that C satisfies (a). Then, $C(2) = C(3) = 0$ and $C(4) = E_1^*$, so $C(4)^2 = -\frac{15b-22}{30b-59}$, hence $C(1)^2 = C^2 - C(4)^2 - (\frac{-15}{2(15b-29)} f^* K_S)^2 = -\frac{1}{2}$.

Assume that C satisfies (b). Then, $C(2) = -\frac{1}{3}B, C(3) = 0$ and $C(4) = E_2^*$, so $C(1)^2 = -1 + \frac{1}{3} + \frac{20b-36}{30b-59} - (\frac{-10}{2(15b-29)} f^* K_S)^2 = 0$. Hence $C(1) = 0$.

Assume that C satisfies (c). Then, $C(2) = 0, C(3) = -\frac{1}{5}B_2$ and $C(4) = E_4^*$, so $C(1)^2 = -1 + \frac{1}{5} + \frac{24b-46}{30b-59} - (\frac{-6}{2(15b-29)} f^* K_S)^2 = 0$. Hence $C(1) = 0$. □

Similarly, we see that there are three, mutually disjoint, (-1)-curves C_1, C_2, C_3 satisfying (a), (b), (c) from Claim 4.8.1, respectively. □

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