ARITHMETIC PROPERTIES OF TRACES OF SINGULAR MODULI ON CONGRUENCE SUBGROUPS

SOON-YI KANG AND CHANG HEON KIM

Abstract. After Zagier proved that the traces of singular moduli are Fourier coefficients of a weakly holomorphic modular form, various arithmetic properties of the traces of singular values of modular functions mostly on the full modular group have been found. The purpose of this paper is to generalize the results for modular functions on congruence subgroups with arbitrary level.

1. Introduction

Singular moduli are special values of the classical modular invariant $j(z)$ at imaginary quadratic arguments in the upper half plane $\mathbb{H}$ which play important roles in number theory. Recently, two papers by Borcherds [3] and Zagier [22] have inspired many works on connecting the traces of singular moduli to the Fourier coefficients of weakly holomorphic modular forms or Harmonic Maass forms of half-integral weight. (See [20] for the list of references.)

To describe the works of Borcherds and Zagier, we begin with the definition of the modular trace of a weakly holomorphic modular function. In this paper, D is always a positive integer congruent to 0 or 3 modulo 4. For each non-square D, we denote by $\mathcal{Q}_{D,N}$ the set of positive definite integral binary quadratic forms

$$Q(x,y) = [Na,b,c] = Nax^2 + bxy + cy^2$$

with discriminant $-D = b^2 - 4Nac$. The set $\mathcal{Q}_{D,N}$ is invariant under the action of the congruence subgroup $\Gamma_0(N) \subseteq \Gamma(1) := PSL_2(\mathbb{Z})$. For a fixed solution $\beta \pmod{2N}$ of $\beta^2 \equiv -D \pmod{4N}$, a smaller set $\mathcal{Q}_{D,N,\beta} = \{[Na,b,c] \in \mathcal{Q}_{D,N} \mid b \equiv \beta \pmod{2N}\}$ is also invariant under the action of $\Gamma_0(N)$, and there is a canonical bijection between $\mathcal{Q}_{D,N,\beta}/\Gamma_0(N)$ and $\mathcal{Q}_{D,1}/\Gamma(1)$ when the discriminant D is not divisible by the square of any prime divisor of N [12]. For each quadratic form Q in $\mathcal{Q}_{D,N,\beta}$, the corresponding Heegner point on the modular curve $X_0(N)$ is the unique root in $\mathbb{H}$ of $Q(x,1) = 0$,

$$z_Q = \frac{-b + i\sqrt{D}}{2Na} \in \mathbb{H}.$$

Denoting the stabilizers of Q in $\Gamma_0(N)$ by $\Gamma_0(N)_Q$, we define the trace of a weakly holomorphic modular function f on $\Gamma_0(N)$ by

$$\mathbf{t}_f(D) = \sum_{Q \in \mathcal{Q}_{D,N,\beta}/\Gamma_0(N)} \frac{1}{|\Gamma_0(N)_Q|} f(z_Q). \tag{1.1}$$

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In particular, the class number $H_N(D)$ is given by

$$H_N(D) = \mathbf{t}_1(D) = \sum_{Q \in \mathcal{Q}_{D,N,\beta}/\Gamma_0(N)} \frac{1}{|\Gamma_0(N)_Q|} \tag{1.2}$$

which is the Hurwitz-Kronecker class number $H(D)$ when $N = 1$.

We define a generalized Hilbert class polynomial $\mathcal{H}_D$ by

$$\mathcal{H}_D(X) = \prod_{Q \in Q_{D,1}/\Gamma(1)} (X - j(z_Q))^{1/|\Gamma(1)_Q|} \tag{1.3}$$

that reduces to the Hilbert class polynomial when $-D < 0$ is a fundamental discriminant. If $q = e^{2\pi iz}$ and $j_1(z) = j(z) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots$ denotes the normalized Hauptmodul for $\Gamma(1)$, then the q-expansion of the polynomial is given by [22, Eq. (11)]

$$\mathcal{H}_D(j(z)) = q^{-H(D)}(1 - \mathbf{t}_{j_1}(D)q + O(q^2)). \tag{1.4}$$

In [3, p. 204], [22, Theorem 3], Borcherds proved that

$$\mathcal{H}_D(j(z)) = q^{-H(D)} \prod_{n=1}^{\infty} (1 - q^n)^{A(n^2,D)}, \tag{1.5}$$

where $A(d, D)$ are the Fourier coefficients of a weight $1/2$ weakly holomorphic modular form. More precisely, for $D \geq 0$, the set of the functions

$$f_D(z) = q^{-D} + \sum_{\substack{d>0 \\ d \equiv 0,3 \pmod 4}} A(d, D)q^d$$

forms a unique basis of the space of weakly holomorphic modular forms of weight $1/2$. Comparing equation (1.4) with (1.5), we see that $\mathbf{t}_{j_1}(D) = A(1, D)$ for all $D > 0$. On the other hand, Zagier showed that

$$g_d(z) = q^{-d} - \sum_{\substack{D \geq 0 \\ D \equiv 0,3 \pmod 4}} A(d, D)q^D$$

is a weakly holomorphic modular form of weight $3/2$ and the set $\{g_d | d > 0\}$ forms a basis of the space of weakly holomorphic modular forms of weight $3/2$.

Making use of this duality relation between $f_D(z)$ and $g_d(z)$, Ahlgren and Ono [1] showed that $\mathbf{t}_{j_1}(p^2D) \equiv 0 \pmod p$ for an odd prime p that splits in $\mathbb{Q}(\sqrt{-D})$ and later Osburn [21] generalized it to a divisibility property of the traces of CM values of Hauptmodul with prime level of genus zero. In [8, Theorem 1.4], the authors with Choi and Jeon established a Treneer type congruence relation for the traces of singular moduli with arbitrary level.

A similar duality relation involving traces of the values of Niebur-Poincaré series on $\Gamma(1)$ was discovered by Bringmann and Ono [4, Theorems 1.1 and 1.2]. For a non-negative integer m and complex numbers s and $z = x + iy$ with $y > 0$, we define the weight zero mth Niebur-Poincaré series on a congruence subgroup Γ by

$$\mathcal{F}_m(z, s) = \sum_{M \in \Gamma_\infty \backslash \Gamma} e(-m\mathrm{Re}Mz)(\mathrm{Im}Mz)^{1/2} I_{s-1/2}(2\pi m \mathrm{Im}Mz), \tag{1.6}$$

where $\Gamma_\infty \subset \Gamma$ is the subgroup of translations, $e(z) = e^{2\pi iz}$, and $I_{s-1/2}$ is the modified Bessel function of the first kind. Niebur [19] observed that every modular function on $\Gamma(1)$ that is holomorphic away from the cusp at infinity can be written as a linear sum of these Poincaré series. This property was utilized by Duke [10] to obtain explicit formulas for the traces of CM values of Hecke type Faber polynomials $j_m(z)$ in terms of Kloosterman sums and the class number $H(D)$. Applying parallel arguments to a modular function with prime level p, the authors with Choi and Jeon [9] derived exact formulas for the traces of singular values on $\Gamma_0(p)$.

As the brief review above reveals, most of the published results on arithmetic properties of the traces of singular moduli hold for the full modular group or $\Gamma_0(p)$ for p prime, and this motivated us to study results presented in this short note. More specifically, with the goal to extend the results for the congruence subgroup $\Gamma_0(N)$ for arbitrary N, we first establish exact formulas for traces of singular values of any modular function that is holomorphic away from the cusp at infinity and then use these to prove a duality relation and a divisibility property of the traces. As for the organization of this note, we present our results in the next section and summarize the proofs for our theorems in the following three sections.

2. Statements of results

2.1. **Exact formulas for traces of singular moduli on congruence subgroups.** Throughout the rest of the paper, Γ always denotes the group $\Gamma_0^*(N)$ generated by $\Gamma_0(N)$ and all Atkin-Lehner involutions. Then the set $\mathcal{Q}_{D,N}$ is invariant under the action of Γ and we can define the trace of a weakly holomorphic modular function with respect to Γ as

$$\mathbf{t}_f^*(D) = \sum_{Q\in\mathcal{Q}_{D,N}/\Gamma} \frac{1}{|\Gamma_Q|} f(z_Q).$$

We denote $H_N^*(D)$ by the corresponding class number, i.e., $\mathbf{t}_1^*(D)$. It is easy to see that if a is the number of prime divisors p of (β, N) such that $p \nmid \dfrac{N}{(\beta, N)}$, then

$$\mathbf{t}_f^*(D) = \frac{1}{2^a}\mathbf{t}_f(D). \tag{2.1}$$

Let us pause for a moment to note on the Poincaré series representation of a weakly holomorphic modular function. The Niebur-Poincaré series $\mathcal{F}_m(z, s)$ on Γ in (1.6) converges absolutely for $\operatorname{Re} s > 1$ and can be analytically continued to the entire s plane and it has no poles at $\operatorname{Re}(s) = 1$ [19, Theorem 5]. Moreover, it is an eigenfunction for the hyperbolic Laplacian $\triangle = -y^2(\partial_x^2 + \partial_y^2)$ with eigenvalue $s(1-s)$ so that $\mathcal{F}_m(z, 1)$ is annihilated by the Laplacian $\triangle$ and thus almost holomorphic on $\mathbb{H}$. Niebur showed that any modular function on $\Gamma(1)$ that is holomorphic away from the cusp at infinity is a linear combination of the $\mathcal{F}_m(z, 1)$ on $\Gamma(1)$ [19, Theorem 6]. Using the similar argument in [9, p. 4], we can prove that it is also true for any modular function f on Γ that is holomorphic away from the cusp at infinity so that we may write

$$f(z) = \sum_{m=1}^{\ell} a_m(2\pi\sqrt{m})\mathcal{F}_m(z, 1) + c_m, \tag{2.2}$$

where $-c_m$ is the constant term in $(2\pi\sqrt{m})\mathcal{F}_m(z,1)$. For example, a Faber polynomial $j_m(z) = (j-744)|_{T_m}$ on $\Gamma(1)$ satisfies

$$j_m(z) = 2\pi\sqrt{m}\mathcal{F}_m(z,1) - 24\sigma(m),$$

where $|_{T_m}$ is the weight 0 Hecke operator and $\sigma(m)$ is the divisor function [22, 10]. If N is a prime p and $p^\alpha||m$, then we have [9]

$$f(z) = \sum_{m=1}^{\ell} a_m \left(2\pi\sqrt{m}\mathcal{F}_m(z,1) - 24\left(\frac{-p^{\alpha+1}}{p+1}\sigma(m/p^\alpha) + \sigma(m)\right)\right).$$

The constant c_m for arbitrary level N can be determined explicitly using properties of Ramanujan sum.

Theorem 2.1. *Let $\mathcal{F}_m(z,s)$ be the Poincaré series defined in (1.6). Then the constant term $-c_m$ in $(2\pi\sqrt{m})\mathcal{F}_m(z,1)$ is given by*

$$-c_m = 24\frac{m}{m_N}\sigma(m_N)\frac{\sum_{e||N}\frac{1}{e}\prod_{p|\frac{N}{e}}\delta_e(p)p^{1-\beta_{e,p}}\left((1-p^{\beta_{e,p}-\alpha_p-2})(1+p^{-1})-1\right)}{\prod_{p|N}(1-p^{-2})}. \tag{2.3}$$

Here $\alpha_p := ord_p(m)$, $\beta_{e,p} = ord_p(\frac{N}{e})$, m_N denotes the largest exact divisor of m satisfying $(m_N, N) = 1$, and $\delta_e(p)$ is defined by

$$\delta_e(p) = \begin{cases} 1, & \text{if } \beta_{e,p} \le \alpha_p + 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

In particular, if m is relatively prime to N, then

$$c_m = -24\sigma(m)\frac{\sum_{e||N}\mu(N/e)e}{N^2\prod_{p|N}(1-p^{-2})}, \tag{2.5}$$

where $\mu(n)$ is the Möbius function. This implies that

$$c_m = \sigma(m)c_1 \qquad \text{whenever} \qquad (m,N) = 1. \tag{2.6}$$

As a consequence of (2.2) and Theorem 2.1, we obtain the exact formulas for the traces of singular values of f.

Theorem 2.2. *Suppose that f is a modular function for Γ whose poles are supported only at ∞ and the principal part is given by $\sum_{m=1}^{\ell} a_m e(-mz)$. Then*

$$\boldsymbol{t}_f^*(D) = \sum_{m=1}^{\ell} a_m \left[c_m H_N^*(D) + \sum_{\substack{c>0 \\ c\equiv 0\,(mod\, 4N)}} S_D(m,c)\sinh\left(\frac{4\pi m\sqrt{D}}{c}\right)\right], \tag{2.7}$$

where $-c_m$ is given in (2.3) and

$$S_D(m,c) = \sum_{x^2\equiv -D (mod\, c)} e(2mx/c) \quad \text{for any positive integers } m \text{ and } c. \tag{2.8}$$

Example 1. Consider

$$j_{45}^* = -1 + \left(\frac{\eta(3z)^2\eta(15z)^2}{\eta(z)\eta(5z)\eta(9z)\eta(45z)}\right),$$

where $\eta(z)$ is the Dedekind eta function defined by $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. Then j_{45}^* is the Hauptmodul for $\Gamma_0^*(45)$ which is of genus 0 and has a Fourier expansion of the form $q^{-1}+0+O(q)$. Let $D=20$. Since the representatives for $\mathcal{Q}_{20,45,40}/\Gamma_0(45)$ are given by $[405,40,1]$ and $[90,-50,7]$, we find from equations (2.1), (2.5) and (2.7) that

$$j_{45}^*(z_{[405,40,1]}) + j_{45}^*(z_{[90,-50,7]}) = -1 + 2\sum_{\substack{c>0\\ c\equiv 0 \pmod{180}}} S_{20}(1,c)\sinh\left(\frac{8\pi\sqrt{5}}{c}\right).$$

The left hand side of the equation is known to be -3, and hence the exponential sum on the right hand side has the value -1.

2.2. **Duality and Divisibility for traces of singular moduli on congruence subgroups.** For more arithmetic properties of traces of singular values of Niebur-Poincarè series and modular functions holomorphic away from the cusp at infinity, we make the Kloosterman sum representation of traces of the Niebur-Poincarè series at Heegner points. In order to define Kloosterman sum, we need the extended Kronecker symbol $\left(\frac{c}{d}\right)$ and

$$\varepsilon_d := \begin{cases} 1, & \text{if } d\equiv 1 \pmod 4,\\ i, & \text{if } d\equiv 3 \pmod 4 \end{cases}$$

that is defined for odd d. For $c,m,n,\lambda\in\mathbb{Z}$ with $c\equiv 0 \pmod 4$, the weight $k:=\lambda+1/2$ Kloosterman sum $K_\lambda(m,n,c)$ is defined by

$$K_\lambda(m,n,c) := \sum_{v \pmod{c}^*}\left(\frac{c}{v}\right)\varepsilon_v^{2\lambda+1} e\left(\frac{m\bar{v}+nv}{c}\right), \tag{2.9}$$

where the sum runs through the primitive residue classes modulo c and $v\bar{v}\equiv 1 \pmod c$.

Following the method developed in [14], [5], [7], [4] and [18], we construct a half integral weight Maass-Poincaré series for arbitrary level $4N$ whose holomorphic coefficients are represented by the Kloosterman sums: For $s\in\mathbb{C}$ and $y\in\mathbb{R}-\{0\}$, we define

$$\mathcal{M}_s(y) := |y|^{-k/2} M_{\frac{k}{2}\mathrm{sgn}(y), s-\frac{1}{2}}(|y|),$$

where $M_{\nu,\mu}$ is the usual M-Whittaker function. And for $m\geq 1$ with $(-1)^{\lambda+1}m\equiv 0,1 \pmod 4$, we define

$$\varphi_{-m,s}(z) := \mathcal{M}_s(-4\pi m y)e(-mx).$$

With these notations, we define the Poincaré series for $\mathrm{Re}(s)>1$ by

$$\mathfrak{P}_{\lambda,n}(-m,s;z) := \sum_{M\in\Gamma_\infty\backslash\Gamma_0(4N)} (\varphi_{-m,s}|_k M)(z)$$ where $|_k$ is the usual weight k slash operator.

Now as in [4] and [18], we apply Kohnen's projection operator [16, p. 250] pr_λ to $\mathfrak{P}_{\lambda,n}(-m,s;z)$ to obtain a new family of weak Maass forms.

$$\mathcal{P}_{\lambda,N}(-m,z) := \begin{cases} \frac{3}{2}\mathfrak{P}_{\lambda,n}(-m,\frac{k}{2};z)|\mathrm{pr}_\lambda, & \text{if } \lambda \geq 1; \\ \frac{3}{2(1-k)\Gamma(1-k)}\mathfrak{P}_{\lambda,n}(-m,1-\frac{k}{2};z)|\mathrm{pr}_\lambda, & \text{if } \lambda \leq 0. \end{cases}$$

The series $\mathcal{P}_{\lambda,N}(-m,z)$ is a weakly holomorphic modular form of weight $\lambda+1/2$ and level $4N$ satisfying Kohnen plus-condition if $\lambda>1$ and it is a weak Maass form if $\lambda \leq 1$ that has Fourier expansion

$$\mathcal{P}_{\lambda,N}(-m,z) = q^{-m} + \sum_{\substack{n\geq 0 \\ (-1)^\lambda n \equiv 0,1 \pmod 4}} b_{\lambda,N}(-m,n)q^n + \mathcal{P}^-_{\lambda,N}(-m,z), \tag{2.10}$$

where $\mathcal{P}^-_{\lambda,N}(-m,z)$ is the non-holomorphic part. It follows from [18, Theorem 2.1] that if m,n are positive integers such that $(-1)^{\lambda+1}m, (-1)^\lambda n \equiv 0,1 \pmod 4$ and N is odd, then the Fourier coefficients $b_{\lambda,N}(-m,n)$ of the weak Maass form $\mathcal{P}_{\lambda,N}(-m,z)$ is given by

$$\begin{aligned} b_{\lambda,N}(-m,n) : &= (-1)^{[\frac{\lambda+1}{2}]}\pi\sqrt{2}(n/m)^{\frac{2\lambda-1}{4}}(1-(-1)^\lambda i) \\ &\quad \times \sum_{\substack{c>0 \\ 4N|c}} \frac{K_\lambda(-m,n,c)}{c}\delta_o(c/4)I_{\lambda-1/2}\left(\frac{4\pi\sqrt{nm}}{c}\right), \end{aligned} \tag{2.11}$$

where $\delta_o(d)=2$ if d is odd and 1 otherwise.

Using the Bruinier-Funke theta lift as in [8], one can construct a weight 3/2 Harmonic weak Maass form $G_N(z)$ whose holomorphic part is $\sum_D H^*_N(D)q^D$. Let

$$\mathcal{P}^*_{1,N}(-m,z) := \mathcal{P}_{1,N}(-m,z) + (-c_1)\delta_\Box(m)G_N(z). \tag{2.12}$$

In addtion, we define

$$\mathcal{P}^*_{0,N}(-m,z) := \mathcal{P}_{0,N}(-m,z) + c_1 H^*_N(m)\theta(z)/2, \tag{2.13}$$

where $\theta(z) = \sum_{n\in\mathbb{Z}} q^{n^2}$ is the Jacobi theta series. Then by generalizing the duality relation $b_{\lambda,1}(-m,n) = -b_{1-\lambda,1}(-n,m)$ due to Bringmann and Ono [4, Theorem 1.1], we can establish the following duality relation for the holomorphic coefficients of $\mathcal{P}^*_{1,N}(-m,z)$ and $\mathcal{P}^*_{0,N}(-m,z)$.

Theorem 2.3. *Let $\mathcal{P}^*_{1,N}(-m,z)$ and $\mathcal{P}^*_{0,N}(-m,z)$ be weak Maass forms defined in (2.12) and (2.13), respectively, with N odd. Assume for $i=0,1$ that*

$$\mathcal{P}^*_{i,N}(-m,z) = \sum_{n\geq 0} B_{i,N}(-m,n)q^n + \text{non holomorphic part.} \tag{2.14}$$

If m is a positive integer that is a square modulo $4N$, then for every positive integer n with $-n \equiv \Box \pmod{4N}$, we have

$$B_{1,N}(-m,n) = -B_{0,N}(-n,m).$$

It is straightforward to show that the trace of the values of Niebur-Poincaré series at Heegner points is a linear sum of $B_{1,N}(-m,n)$ using the exact formula for traces of singular moduli in Theorem 2.2. We will elaborate on this in Lemma 4.1 below which leads to a divisibility property of the traces of singular moduli that is a generalization of the results in [1, 11, 13, 15, 21].

For an integer $m \geq 1$, the Hecke operators $T(m)$ defined in [22, 13] act on the space of weak Maass forms (see [20, p.35]). We denote by $B_{0,N}^{(m)}(-D,n)$ the coefficient of q^n in $\mathcal{P}_{0,N}^{*}(-D,z)|_{1/2}T(m)$.

Theorem 2.4. *Let $-D$ be a fundamental discriminant that is a square modulo $4N$. Suppose that f is a modular function for Γ with odd level N which is holomorphic away from the cusp at infinity. If f has the principal part $\sum_{m=1}^{\ell} a_m e(-mz)$ at ∞ with $a_m \in \mathbb{Z}$ such that for all non-zero a_m, $(m,N)=1$, then for every prime p for which $(p,N)=1$ and $\left(\frac{-D}{p}\right)=1$,*

$$\mathbf{t}_f(p^{2n}D) = p^n \sum_{m=1}^{\ell} a_m B_{0,N}^{(m)}(-D,p^{2n}). \tag{2.15}$$

If we assume a modular function $f \in \mathbb{Q}((q))$ satisfies the condition in Theorem 2.4 and both trace of f and the sum $\sum_m a_m B_{0,N}^{(m)}(-D,p^{2n})$ are integers, we have the congruence

$$\mathbf{t}_f(p^{2n}D) \equiv 0 \pmod{p^n}.$$

Example 2. Consider

$$f = \left(\frac{\eta(z)}{\eta(37z)}\right)^2 - 2 + 37\left(\frac{\eta(37z)}{\eta(z)}\right)^2.$$

Then f is a modular function for $\Gamma_0^*(37)$ which is of genus 1 and has a Fourier expansion of the form $q^{-3}-2q^{-2}-q^{-1}+0+O(q)$. When $D=11$, the computation from Theorem 2.1, Lemma 3.1 and Theorem 2.2 shows that $\mathbf{t}_f(11)=\mathbf{t}_3(11)-2\mathbf{t}_2(11)-\mathbf{t}_1(11)=5$ is an integer while $\mathbf{t}_1(11)$, $\mathbf{t}_2(11)$, $\mathbf{t}_3(11)$ are not integers. Furthermore, $\mathbf{t}_f(p^{2n}\cdot 11) \equiv 0 \pmod{p^n}$ for a prime p for which $\left(\frac{-11}{p}\right)=1$. For example, for the first three of such primes 3, 5 and 23, we have

$$\begin{aligned}
\mathbf{t}_f(99) &= \mathbf{t}_f(3^2\cdot 11) = -6 \equiv 0 \pmod{3},\\
\mathbf{t}_f(275) &= \mathbf{t}_f(5^2\cdot 11) = -75 \equiv 0 \pmod{5},\\
\mathbf{t}_f(5819) &= \mathbf{t}_f(23^2\cdot 11) = 246920364 \equiv 0 \pmod{23},\\
\mathbf{t}_f(891) &= \mathbf{t}_f(3^4\cdot 11) = 2988 \equiv 0 \pmod{9},\\
\mathbf{t}_f(6875) &= \mathbf{t}_f(5^4\cdot 11) = -1966145550 \equiv 0 \pmod{25},\\
\mathbf{t}_f(8019) &= \mathbf{t}_f(3^6\cdot 11) = -15195121128 \equiv 0 \pmod{27}.
\end{aligned}$$

3. Proofs of Exact Formulas for Traces

In this section, we will give the proofs of the explicit formulas for the traces of singular moduli in Theorems 2.1 and 2.2. We first establish an explicit formula for the traces of the values of Niebur-Poincarè series at Heegner points without a proof as it can be derived in a very similar way to [9, Lemma 3].

Lemma 3.1. *Let $\mathcal{F}_m^*(z,s) = (2\pi\sqrt{m})\mathcal{F}_m(z,s)+c_m$, where $\mathcal{F}_m(z,s)$ is defined in (1.6) and $-c_m$ is the constant term in $(2\pi\sqrt{m})\mathcal{F}_m(z,1)$ which is explicitly given in (2.3). Then the trace of the* *values of $\mathcal{F}_m^*$ at Heegner points is given by*

$$\boldsymbol{t}_m(D) := \sum_{Q\in\mathcal{Q}_{D,N}/\Gamma} \frac{1}{|\Gamma_Q|}\mathcal{F}_m^*(z_Q,1) = c_m H_N^*(D) + \sum_{\substack{c>0\\ c\equiv 0\,(mod\,4N)}} S_D(m,c)\sinh\left(\frac{4\pi m\sqrt{D}}{c}\right). \tag{3.1}$$

Theorem 2.2 follows immediately from Lemma 3.1 and (2.2), and thus it only remains to evaluate the constant term c_m given in Theorem 2.1 in order to complete the proof of exact formulas for the traces of singular values of a weakly holomorphic modular function on Γ.

Proof of Theorem 2.1. First, we recall that the Niebur-Poincaré series $\mathcal{F}_m(z,s)$ in (1.6) has the following Fourier expansion [19, Theorem 1]; for $\operatorname{Re} s > 1$,

$$\mathcal{F}_m(z,s) = e(-mx)y^{1/2}I_{s-1/2}(2\pi m y) + \sum_{n=-\infty}^{\infty} b_n(y,s;-m)e(nx), \tag{3.2}$$

where $b_n(y,s;-m) \to 0$ $(n\neq 0)$ exponentially as $y\to\infty$. Then the constant term in $(2\pi\sqrt{m})\mathcal{F}_m(z,1)$ is

$$\begin{aligned} -c_m &= \lim_{s\to 1} 2\pi\sqrt{m}b_0(y,s,-m) \\ &= \lim_{s\to 1} 2\pi\sqrt{m}(2\pi^s m^{s-1/2}\phi_m(s)/\Gamma(s))y^{1-s}/(2s-1) = 4\pi^2 m \lim_{s\to 1}\phi_m(s). \end{aligned} \tag{3.3}$$

Here $\phi_m(s) = \sum_{c>0} S(m,0;c)c^{-2s}$ and $S(m,n;c)$ is the general Kloosterman sum $\sum_{0\le d<|c|} e((ma+nd)/c)$ for $\left(\begin{smallmatrix} a & * \\ c & d\end{smallmatrix}\right)\in\Gamma$. As $M\in\Gamma_0^*(N)$ if and only if M is of the form $\begin{pmatrix}\sqrt{e}\mathfrak{a} & \mathfrak{b}/\sqrt{e}\\ N\mathfrak{c}/\sqrt{e} & \sqrt{e}\mathfrak{d}\end{pmatrix}_{\det=1}$ for some $e\|N$ with $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}\in\mathbb{Z}$ and $(\mathfrak{a},N\mathfrak{c}/e)=1$, we can identify the sum $S(m,0;c) = \sum_{0\le d<|c|} e(ma/c)$ with the sum of m-th powers of primitive $N\mathfrak{c}/e$-th roots of unity. Hence if we denote the Ramanujan sum by $u_n(q)$, that is the sum of n-th powers of primitive q-th roots of unity, we find that

$$\begin{aligned} \phi_m(s) &= \sum_{c>0} S(m,0;c)c^{-2s} = \sum_{e\|N}\sum_{\substack{\mathfrak{c}\ge 1\\ (e,\mathfrak{c})=1}} u_m(N\mathfrak{c}/e)\frac{N^{-2s}}{e^{-s}}\mathfrak{c}^{-2s} \\ &= \sum_{e\|N} e^{-s}\sum_{\substack{\mathfrak{c}\ge 1\\ (e,\mathfrak{c})=1}} u_m(N\mathfrak{c}/e)(N\mathfrak{c}/e)^{-2s} \\ &= \sum_{e\|N} e^{-s}\left(\prod_{p|\frac{N}{e}}\sum_{k\ge ord_p(\frac{N}{e})} u_m(p^k)p^{-2sk}\prod_{p\nmid N}\sum_{k\ge 0} u_m(p^k)p^{-2sk}\right), \end{aligned} \tag{3.4}$$

where the last equality holds due to the multiplicative property of $u_n(q)$ as a function of q and the fact $(e,\mathfrak{c})=1$. Recall the following known fact on the Ramanujan sum:

$$u_n(p^k) = \begin{cases} 0, & \text{if } p^{k-1}\nmid n,\\ -p^{k-1}, & \text{if } p^{k-1}||n,\\ \varphi(p^k), & \text{if } p^k\mid n,\end{cases} \tag{3.5}$$

where $\varphi(n)$ is Euler's totient function. Letting $\alpha = \alpha_p = ord_p(m)$ and m_N be the largest exact divisor of m satisfying $(m_N, N) = 1$, we deduce from (3.5) that

$$\begin{aligned}\prod_{p\nmid N}\sum_{k\geq 0} u_m(p^k)p^{-2sk} &= \prod_{p\nmid N}(1+\varphi(p)p^{-2s}+\varphi(p^2)p^{-4s}+\cdots+\varphi(p^\alpha)p^{-2\alpha s}-p^\alpha p^{-2(\alpha+1)s}) \\ &= \prod_{p\nmid N}(1-p^{-2s})(1+p^{1-2s}+\cdots+(p^{1-2s})^\alpha) \\ &= \frac{\zeta(2s)^{-1}}{\prod_{p\mid N}(1-p^{-2s})}\sigma_{1-2s}(m_N).\end{aligned} \tag{3.6}$$

Also, if $\beta = \beta_{e,p} = ord_p(\frac{N}{e})$ and $\delta_e(p)$ is defined as in (2.4), then we obtain from (3.5) that

$$\begin{aligned}\prod_{p\mid \frac{N}{e}}\sum_{k\geq \beta} u_m(p^k)p^{-2sk} &= \prod_{p\mid\frac{N}{e}}\delta_e(p)\left(\varphi(p^\beta)p^{-2\beta s}+\varphi(p^{\beta+1})p^{-2(\beta+1)s}+\cdots+\varphi(p^\alpha)p^{-2\alpha s}-p^\alpha p^{-2(\alpha+1)s}\right) \\ &= \prod_{p\mid\frac{N}{e}}\delta_e(p)(p^{1-2s})^{\beta-1}\left(\frac{1-(p^{1-2s})^{\alpha-\beta+2}}{1-p^{1-2s}}(1-p^{-2s})-1\right).\end{aligned} \tag{3.7}$$

Therefore, the theorem follows from (3.3), (3.4), (3.6), and (3.7). □

4. Duality for Traces

We first prove the Kloosterman sum representation of the trace of the values of Niebur-Poincaré series at Heegner points and then the duality relation given in Theorem 2.3.

Lemma 4.1. *Let $\boldsymbol{t}_m(D)$ be the trace of the Poincaré series of odd level N at Heegner points in (3.1). If $(m, N) = 1$, then*

$$\boldsymbol{t}_m(D) = -\sum_{\nu\mid m}\nu B_{1,N}(-\nu^2, D),$$

where $B_{1,N}(-m, n)$ is given by

$$B_{1,N}(-m,n) = -c_1\delta_\square(m)H_N^*(n) + b_{1,N}(-m,n). \tag{4.1}$$

Here $-c_1$ is the constant given in (2.5) with $m = 1$, the function $\delta_\square(m) = 1$ if m is a square and zero otherwise, and $b_{1,N}(-m, n)$ is given in (2.11) with $\lambda = 1$.

Proof. From Kohnen's result on the relation between Kloosterman sum and Salié sum given in [16, Proposition 5] and [18, Proposition 2.2], we can write the sum in the far right side in (3.1) as

$$\begin{aligned}\sum_{\substack{c>0\\ c\equiv 0\,(\mathrm{mod}\,4N)}} S_D(m,c)\sinh\left(\frac{4\pi m\sqrt{D}}{c}\right) &= \sum_{\substack{c>0\\ 4N\mid c}}\sum_{d\mid(\frac{c}{4},m)}(1+i)(c/d)^{-1/2}\delta_o(c/4d) \\ &\times K_1(-\frac{m^2}{d^2},D,c/d)\left(\frac{2m\pi^2\sqrt{D}}{c}\right)^{1/2} I_{1/2}\left(\frac{4\pi m\sqrt{D}}{c}\right).\end{aligned} \tag{4.2}$$

Since $(m, N) = 1$ and $c_m = \sigma(m)c_1$ when $(m, N) = 1$, we deduce from (3.1) and (4.2) that

$$\mathbf{t}_m(D) = c_1 H_N^*(D)\sum_{\nu|m}\nu \quad + \quad \sum_{\nu|m}\sum_{\substack{c>0\\ 4\nu N|c}}(1+i)(c/\nu)^{-1/2}\delta_o(c/4\nu)$$

$$\times \quad K_1(-\frac{m^2}{\nu^2}, D, c/\nu)\left(\frac{2m\pi^2\sqrt{D}/\nu}{c/\nu}\right)^{1/2} I_{1/2}\left(\frac{4\pi m\sqrt{D}/\nu}{c/\nu}\right). \tag{4.3}$$

Replacing c/ν by c and m/ν by ν, we complete the proof. □

We note that $B_{1,N}(-m, n)$ in Lemma 4.1 are holomorphic coefficients of $\mathcal{P}^*_{1,N}(-m, z)$.

Proof of Theorem 2.3. According to the Fourier development of $\mathcal{P}_{\lambda,N}(-m, z)$ computed in [18, Theorem 2.1], the coefficient $b_{1,N}(-m, n)$ in the holomorphic part of $\mathcal{P}_{1,N}(-m, z)$ is given by

$$b_{1,N}(-m,n) = -\pi\sqrt{2}(n/m)^{1/4}(1+i)\sum_{\substack{c>0\\ 4N|c}}\delta_o(c/4)\frac{K_1(-m,n,c)}{c}I_{1/2}\left(\frac{4\pi\sqrt{nm}}{c}\right).$$

Applying [4, Proposition 3.1], we may write the right-hand side as

$$-\pi\sqrt{2}(n/m)^{1/4}(1+i)\sum_{\substack{c>0\\ 4N|c}}\delta_o(c/4)(-i)\frac{K_0(-m,n,c)}{c}I_{1/2}\left(\frac{4\pi\sqrt{nm}}{c}\right),$$

which is the Fourier coefficients $-b_{0,N}(-n, m)$ in the holomorphic part of $\mathcal{P}_{0,N}(-m, z)$ by [18, Theorem 2.1]. Thus

$$b_{1,N}(-m,n) = -b_{0,N}(-n,m). \tag{4.4}$$

By the definition of $B_{\lambda,N}(-m, n)$, we have

$$B_{1,N}(-m,n) = -c_1\delta_{\square,m}H_N^*(n) + b_{1,N}(-m,n) \tag{4.5}$$

and

$$B_{0,N}(-m,n) = c_1\delta_{\square,n}H_N^*(m) + b_{0,N}(-m,n). \tag{4.6}$$

Now the theorem follows from (4.4), (4.5) and (4.6). □

5. Divisibility for Traces

We start the proof of Theorem 2.4 by showing that the Niebur-Poincaré series $\mathcal{F}_m(z, 1)$ is generated by the action of Hecke operator on $\mathcal{F}_1(z, 1)$.

Lemma 5.1. *For every positive integer m relatively prime to N, we have*

$$\mathcal{F}_1^*(z,1)|_{T_m} = \mathcal{F}_m^*(z,1). \tag{5.1}$$

And for a prime p not dividing N,

$$\mathcal{F}_m^*(z,1)|_{T_{p^n}} = \sum_{p^i|(m,p^n)} p^i\mathcal{F}^*_{\frac{p^n m}{p^{2i}}}(z,1) = \sum_{i=0}^{\min(n,s)} p^i\mathcal{F}^*_{p^{n+s-2i}m_0}(z,1), \tag{5.2}$$

where $m = p^s m_0$ with $p \nmid m_0$ and $s \geq 0$.

Proof. As we can easily deduce (5.1) from (5.2), we only prove (5.2). By [2, Lemma 6], [17, Lemmas 2.5 and 2.6] and [17, Theorem 6.3], $\mathcal{F}_m^*(z,1)|_{T_{p^n}}$ is on Γ and has a pole only at ∞. Since $\mathcal{F}_m^*(z,1)$ is harmonic, so is $\mathcal{F}_m^*(z,1)|_{T_{p^n}}$. Now in the case $n=1$, if we compare the principal parts of the functions in both sides of (5.2) and apply [19, Theorem 6], we obtain the result immediately. If we use induction on n as in [17, Theorem 6.2], we get the assertion. ☐

By Lemma 4.1, for m and p satisfying $(m,N)=1$ and $(p,N)=1$, we have

$$\mathbf{t}_{m_0p^j}(D) = -\sum_{t=0}^{j}\sum_{\nu|m_0} p^t\nu B_{1,N}(-p^{2t}\nu^2, D) = \sum_{t=0}^{j}\sum_{\nu|m_0} p^t\nu B_{0,N}(-D, p^{2t}\nu^2)$$

$$= B_{0,N}^{(m_0p^j)}(-D,1). \quad \text{(cf. [22, formula (19)])} \tag{5.3}$$

On the other hand, by the definition of $\mathbf{t}_m(D)$ and (5.2) (cf. [22, Section 6]),

$$\sum_{i=0}^{n}\mathbf{t}_m(p^{2i}D) = \sum_{i=0}^{n}\left(\frac{-D}{p^{n-i}}\right)\mathbf{t}_m(p^{2i}D) = \sum_{i=0}^{\min(s,n)} p^i\mathbf{t}_{p^{n+s-2i}m_0}(D). \tag{5.4}$$

It then follows from (5.3) and (5.4) that

$$\sum_{i=0}^{n}\mathbf{t}_m(p^{2i}D) = \sum_{i=0}^{\min(s,n)} p^i\mathbf{t}_{p^{n+s-2i}m_0}(D) = \sum_{i=0}^{\min(s,n)} p^i B_{0,N}^{(m_0p^{n+s-2i})}(-D,1)$$

$$= \sum_{i=0}^{n} p^i B_{0,N}^{(m_0p^s)}(-D,p^{2i})$$

$$\text{since } \sum_{i=0}^{\min(s,n)} p^iT(m_0p^{n+s-2i}) = T(m_0p^s)T(p^n) \text{ (cf. [13, p. 155]).}$$

Thus we have

$$\sum_{i=0}^{n}\mathbf{t}_m(p^{2i}D) = \sum_{i=0}^{n} p^i B_{0,N}^{(m)}(-D,p^{2i}).$$

Now induction argument in $n \geq 0$ shows that

$$\mathbf{t}_m(p^{2n}D) = p^n B_{0,N}^{(m)}(-D,p^{2n}),$$

which proves (2.15) since $\mathbf{t}_f(p^{2n}D) = \sum_{m=1}^{l} a_m\mathbf{t}_m(p^{2n}D)$.

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Korea Advanced Institute of Science and Technology, Department of Mathematical Sciences, Daejeon, 305-701 Korea
E-mail address: `s2kang@kaist.ac.kr`

Department of Mathematics, Hanyang University, Seoul, 133-791 Korea
E-mail address: `chhkim@hanyang.ac.kr`