ON RAY CLASS ANNIHILATORS OF CYCLOTOMIC FUNCTION FIELDS

SUNGHAN BAE AND HWANYUP JUNG*

Abstract. Let $\mathcal{K}$ be a cyclotomic function field over a global function field k with Galois group G. In this paper we define an ideal $S_{\mathfrak{d}}$ of $R = \mathbb{Z}[G]$ and show that it annihilates the $\mathfrak{d}$-ray class group $\mathcal{C}_{\mathfrak{d}}$ of $\mathcal{K}$. We also investigate the relation between the index $(R^- : S_{\mathfrak{d}}^-)$ and the order of $\mathcal{C}_{\mathfrak{d}}^-$.

1. Introduction

Let $K = \mathbb{Q}(\zeta_n)$ be the n-th cyclotomic field with Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$. Stickelberger introduced an ideal S (called the Stickelberger ideal of K) of $R = \mathbb{Z}[G]$ which annihilates the ideal class group $\mathcal{C}$ of K. In [Si], Sinnott showed that the index of the minus part of S in the minus part of R is equal to the minus class number of K up to a power of 2. For any integer $d \geq 1$, Schmidt ([Sc]) introduced an ideal S_d (called the d-Stickelberger ideal of K) of R which annihilates the d-ray class group $\mathcal{C}_d$ of K and showed that the index of the minus part of S_d in the minus part of R is equal to the order of the minus part of $\mathcal{C}_d$ up to a power of 2. In this paper we consider the analogous problem in function fields. We first introduce some notations.

Let k be a global function field over the finite field $\mathbb{F}_q$ with q elements of characteristic p. Fix a place ∞ of k of degree 1 and fix a sign function $\mathrm{sgn} : k_\infty \to \mathbb{F}_q$ with $\mathrm{sgn}(0) = 0$, where k_∞ is the completion of k at ∞. We call $x \in k$ positive if $\mathrm{sgn}(x) = 1$, and write $x \gg 0$. Let $\mathbb{A}$ be the Dedekind subring of k consisting of the functions regular away from ∞. Let $\mathfrak{e}$ be the unit ideal of $\mathbb{A}$ and $K_{\mathfrak{e}}$ the Hilbert class field of (k, ∞), and $G_{\mathfrak{e}} = \mathrm{Gal}(K_{\mathfrak{e}}/k)$. We denote by T_0 the set of all non-zero integral ideals of $\mathbb{A}$ and $T_0^* = T_0 \backslash \{\mathfrak{e}\}$. For any $\mathfrak{n} \in T_0^*$, we set the followings:

- $K_{\mathfrak{n}}$:= the cyclotomic function field of the triple $(k, \infty, \mathrm{sgn})$ of conductor $\mathfrak{n}$.
- $G_{\mathfrak{n}} := \mathrm{Gal}(K_{\mathfrak{n}}/k)$.
- J := the inertia group at ∞ in $G_{\mathfrak{n}}$, which we call the sign group. Note that J is naturally isomorphic to $\mathbb{F}_q^*$.
- $K_{\mathfrak{n}}^+$:= the fixed field of J, which we call the maximal real subfield of $K_{\mathfrak{n}}$.
- $|A|$:= the cardinality of a set A.
- $\phi(\mathfrak{n}) := |(\mathbb{A}/\mathfrak{n})^*|$ = the number of units in $\mathbb{A}/\mathfrak{n}$.

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*Corresponding author.

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- $s(A) := \sum_{\sigma\in A}\sigma \in \mathbb{Z}[G_\mathfrak{n}]$ for a subset A of $G_\mathfrak{n}$.
- $\varepsilon^- := 1 - s(J)/(q-1) \in \mathbb{Q}[G_\mathfrak{n}]$.

Let $\mathcal{O}_{K_\mathfrak{n}}$ be the integral closure of $\mathbb{A}$ in $K_\mathfrak{n}$. For a non-zero integral ideal $\mathfrak{N}$ of $\mathcal{O}_{K_\mathfrak{n}}$, let $\mathcal{I}_\mathfrak{N}$ be the group of non-zero fractional ideals of $\mathcal{O}_{K_\mathfrak{n}}$ prime to $\mathfrak{N}$ and $\mathcal{P}_{\mathfrak{N},1}$ be the subgroup of $\mathcal{I}_\mathfrak{N}$ consisting of principal ideals (x) satisfying $x \equiv 1 \bmod \mathfrak{N}$. Then $\mathcal{C}_\mathfrak{N} = \mathcal{I}_\mathfrak{N}/\mathcal{P}_{\mathfrak{N},1}$ is called the $\mathfrak{N}$-ray class group of $K_\mathfrak{n}$. For any $\mathfrak{d} \in T_0$, we write $\mathcal{C}_\mathfrak{d} := \mathcal{C}_{\mathfrak{d}\mathcal{O}_{K_\mathfrak{n}}}$ for simplicity. In this paper we define an ideal $S_\mathfrak{d}$ of $R = \mathbb{Z}[G_\mathfrak{n}]$ by using the Stickelberger elements and show that it annihilates the $\mathfrak{d}$-ray class group $\mathcal{C}_\mathfrak{d}$ of $K_\mathfrak{n}$. Our proof relies on the Hayes' proof of Brumer-Stark conjecture for function fields ([Ha]). For any R-module M, set $M^- := \{m \in M : s(J)\cdot m = 0\}$ which we call the minus part of M. We also show that the ℓ-part of the index $(R^- : S_\mathfrak{d}^-)$ is equal to the ℓ-part of $|\mathcal{C}_\mathfrak{d}^-|$ for any prime number ℓ with $\ell \nmid (q-1)$ assuming that $\mathfrak{n}$ is square free if $\ell = p$.

We fix the following notations.

- $h := |G_\mathfrak{e}| =$ the class number of k.
- $N(\mathfrak{a}) := q^{\deg(\mathfrak{a})}$ for any $\mathfrak{a} \in T_0$.
- $(\mathfrak{a},\mathfrak{b}) :=$ the greatest common divisor of $\mathfrak{a}$ and $\mathfrak{b}$ for any $\mathfrak{a},\mathfrak{b} \in T_0$.
- $N(\mathfrak{u}) := N(\mathfrak{a})/N(\mathfrak{b})$ for any non-zero fractional ideal $\mathfrak{u}$ of $\mathbb{A}$, where $\mathfrak{u} = \mathfrak{a}\mathfrak{b}^{-1}$ with $\mathfrak{a},\mathfrak{b} \in T_0$ and $(\mathfrak{a},\mathfrak{b}) = \mathfrak{e}$.
- $\bar{\mathfrak{a}} := \prod_{\mathfrak{p}|\mathfrak{a}} \mathfrak{p}$, where $\mathfrak{p}$ runs over all prime ideals of $\mathbb{A}$ dividing $\mathfrak{a}$.
- For each prime number ℓ, $|\cdot|_\ell$ denotes the normalized ℓ-adic absolute value, i.e., $|\ell|_\ell = 1/\ell$.

From now on we fix $\mathfrak{n} \in T_0^*$ and write $\mathcal{K} := K_\mathfrak{n}, \mathcal{K}^+ := K_\mathfrak{n}^+$ and $G := G_\mathfrak{n}$ for simplicity.

2. Annihilator of Ray Classes

Let $\mathfrak{a},\mathfrak{b} \in T_0$. We say that $\mathfrak{b}$ *is congruent to* $\mathfrak{a}$ *modulo* $\mathfrak{n}$ if there exists $x \in \mathfrak{a}^{-1}\mathfrak{n}$, $1 + x \gg 0$ such that $\mathfrak{b} = (1+x)\mathfrak{a}$, and write $\mathfrak{a} \sim_\mathfrak{n} \mathfrak{b}$. Then $\sim_\mathfrak{n}$ is an equivalence relation on T_0. For more details on the relation $\sim_\mathfrak{n}$, we refer to [Y2].

For $x \in k^*$, write $||x|| := N(x\mathbb{A})$. For $\mathfrak{a} \in T_0$, let $\mathfrak{a}_1 = \mathfrak{a}(\mathfrak{n},\mathfrak{a})^{-1}$ and $\mathfrak{n}_1 = \mathfrak{n}(\mathfrak{n},\mathfrak{a})^{-1}$. We define for $\mathrm{Re}(s) > 1$

$$Z_\mathfrak{n}(s,\mathfrak{a}) := N(\mathfrak{a})^{-s} \sum_{\substack{x\in\mathfrak{a}^{-1}\mathfrak{n}\\ 1+x\gg 0}} ||1+x||^{-s} = N(\mathfrak{n},\mathfrak{a})^{-s}\zeta_{\mathfrak{n}_1}(s,\mathfrak{a}_1),$$

where $\zeta_{\mathfrak{n}_1}(s,\mathfrak{a}_1)$ is the partial zeta function of the class containing $\mathfrak{a}_1$ in the narrow ray class group of $\mathbb{A}$ modulo $\mathfrak{n}_1$. It has meromorphic continuation to the whole complex plane and is holomorphic except for a simple pole at $s = 1$. For $\mathfrak{a},\mathfrak{b} \in T_0$, if $\mathfrak{a} \sim_\mathfrak{n} \mathfrak{b}$, then we have $Z_\mathfrak{n}(s,\mathfrak{a}) = Z_\mathfrak{n}(s,\mathfrak{b})$. It is well known that $(q-1)Z_\mathfrak{n}(0,\mathfrak{a})$ is an integer.

Define

$$\theta_\mathfrak{n} := \sum_{\mathfrak{a} \bmod {}^*\mathfrak{n}} Z_\mathfrak{n}(0,\mathfrak{a})\sigma_\mathfrak{a}^{-1} \in \mathbb{Q}[G],$$

where $\mathfrak{a} \bmod {}^*\mathfrak{n}$ means that the sum is over the representatives of the narrow ray classes of $\mathbb{A}$ modulo $\mathfrak{n}$, and $\sigma_\mathfrak{a}$ is the Artin automorphism associated to the ideal $\mathfrak{a}$. For $\mathfrak{f}|\mathfrak{n}$, define

$$\theta'_\mathfrak{f} := \sum_{\mathfrak{a} \bmod {}^*\mathfrak{n}} Z_\mathfrak{f}(0,\mathfrak{a})\sigma_\mathfrak{a}^{-1} \in \mathbb{Q}[G]$$

and

$$\theta_\mathfrak{f} := \sum_{\mathfrak{a} \bmod {}^*\mathfrak{f}} Z_\mathfrak{f}(0,\mathfrak{a})\sigma_\mathfrak{a}^{-1} \in \mathbb{Q}[G_\mathfrak{f}].$$

Then $\theta'_\mathfrak{f} = \mathrm{Cor}_{\mathcal{K}/K_\mathfrak{f}}(\theta_\mathfrak{f})$ and $\mathrm{Res}_{\mathcal{K}/K_\mathfrak{f}}(\theta'_\mathfrak{f}) = [\mathcal{K} : K_\mathfrak{f}]\theta_\mathfrak{f}$.

Lemma 2.1. *Let $\mathfrak{p}$ be a prime ideal of $\mathbb{A}$ dividing $\mathfrak{n}$ and let $\mathfrak{f} = \mathfrak{n}\mathfrak{p}^{-1}$.*

(i) $\mathrm{Res}_{\mathcal{K}/K_\mathfrak{f}}(\theta_\mathfrak{n}) = \begin{cases} \theta_\mathfrak{f}, & \textit{if } \mathfrak{p}|\mathfrak{f} \\ (1-\sigma_\mathfrak{p}^{-1})\theta_\mathfrak{f}, & \textit{otherwise.} \end{cases}$

(ii) *Let $H = \mathrm{Gal}(\mathcal{K}/K_\mathfrak{f})$. Then*

$$\theta'_\mathfrak{f} = \begin{cases} s(H)\theta_\mathfrak{n}, & \textit{if } \mathfrak{p}|\mathfrak{f}, \\ s(H)\theta_\mathfrak{n} + \mathrm{Cor}_{\mathcal{K}/K_\mathfrak{f}}(\sigma_\mathfrak{p}^{-1}\theta_\mathfrak{f}), & \textit{otherwise.} \end{cases}$$

Here $\sigma_\mathfrak{p}$ is the Artin automorphism associated to $\mathfrak{p}$ in $G_\mathfrak{f}$.

Proof. (i) Corollary 1.7 and Proposition 1.8 of [T]. (ii) follows immediately from (i). □

For any $\mathfrak{c} \in T_0$, define

$$\theta_\mathfrak{n}(\mathfrak{c}) := (\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})})^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}}.$$

Then $\theta_\mathfrak{n} = \theta_\mathfrak{n}(\mathfrak{e})$ and $\theta'_\mathfrak{f} = \theta_\mathfrak{n}(\mathfrak{n}\mathfrak{f}^{-1})$ for $\mathfrak{f}|\mathfrak{n}$. For $\mathfrak{d} \in T_0$, we define

$$\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c}) := \sum_{\mathfrak{a}|\mathfrak{d}} \mu(\mathfrak{a})\frac{N(\mathfrak{d})}{N(\mathfrak{a})}\theta_\mathfrak{n}(\mathfrak{a}\mathfrak{c}),$$

where $\mu(\mathfrak{a})$ is 0 if $\mathfrak{a}$ is not square free, and $(-1)^t$ if $\mathfrak{a}$ is the product of t distinct prime ideals of $\mathbb{A}$. For a prime ideal $\mathfrak{p}$ of $\mathbb{A}$, we have

$$\delta_{\mathfrak{n},\mathfrak{p}}(\mathfrak{c}) = N(\mathfrak{p})\theta_\mathfrak{n}(\mathfrak{c}) - \theta_\mathfrak{n}(\mathfrak{p}\mathfrak{c}) \text{ and } \delta_{\mathfrak{n},\mathfrak{p}^n}(\mathfrak{c}) = N(\mathfrak{p}^{n-1})\delta_{\mathfrak{n},\mathfrak{p}}(\mathfrak{c}) \text{ for } n \geq 1.$$

It is easy to see that if $\mathfrak{a} \sim_\mathfrak{n} \mathfrak{b}$, then $\theta_\mathfrak{n}(\mathfrak{a}) = \theta_\mathfrak{n}(\mathfrak{b})$ and $\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{a}) = \delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{b})$.

We define an R-ideal

$$S_\mathfrak{d} := \Big(\sum_{\mathfrak{c} \bmod \sim_\mathfrak{n}} R \cdot \delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c})\Big) \cap R,$$

where $\mathfrak{c} \bmod \sim_\mathfrak{n}$ means that the sum is over the representatives of the classes of T_0 modulo $\sim_\mathfrak{n}$, and call it the $\mathfrak{d}$-*Stickelberger ideal* of $\mathcal{K}$. Since $\delta_{\mathfrak{n},\mathfrak{e}}(\mathfrak{c}) = (\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})})^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}}$,

$$S_\mathfrak{e} = \Big(\sum_{\mathfrak{c} \bmod \sim_\mathfrak{n}} R \cdot \theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})}\Big) \cap R = \Big(\sum_{\mathfrak{f}|\mathfrak{n}} R \cdot \theta'_\mathfrak{f}\Big) \cap R$$

is the Stickelberger ideal of $\mathcal{K}$ defined by Yin in [Y1].

Proposition 2.2. *If* $\mathfrak{d} \neq \mathfrak{e}$*, then* $\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c}) \in R$ *for all* $\mathfrak{c}$ mod $\sim_{\mathfrak{n}}$.

Proof. Since $(q-1)\theta_{\mathfrak{n}}(\mathfrak{c}) \in R$, it suffices to show that

$$\sum_{\mathfrak{a}|\mathfrak{d}} \mu(\mathfrak{a})\theta_{\mathfrak{n}}(\mathfrak{a}\mathfrak{c}) \in R.$$

Let $S' = \sum_{\mathfrak{f}|\mathfrak{n}} R\cdot\theta'_{\mathfrak{f}}$ and let γ be a fixed generator of $\mathbb{F}_q^*$. The map $\psi : S' \to \mathbb{F}_q^*$ defined by $\psi(\theta) = \gamma^{(q-1)a_1}$, where a_1 is the coefficient of 1 in θ, is a well defined surjective homomorphism with kernel $S' \cap R$ (see the proof of Lemma 4.2 in [ABJ]). Moreover, $\psi(\sigma\theta) = \psi(\theta)$ for any $\theta \in S'$ and $\sigma \in G$. Since $\theta'_{\mathfrak{f}} - N(\mathfrak{n}\mathfrak{f}^{-1})\theta_{\mathfrak{n}} \in R$ for $\mathfrak{f}|\mathfrak{n}$, we have

$$\psi(\theta'_{\mathfrak{f}}) = \psi(\theta_{\mathfrak{n}})^{N(\mathfrak{n}\mathfrak{f}^{-1})} = \psi(\theta_{\mathfrak{n}}).$$

Thus

$$\psi\big(\sum_{\mathfrak{a}|\mathfrak{d}} \mu(\mathfrak{a})\theta_{\mathfrak{n}}(\mathfrak{a}\mathfrak{c})\big) = \psi(\theta_{\mathfrak{n}})^{\sum_{\mathfrak{a}|\mathfrak{d}}\mu(\mathfrak{a})} = 1,$$

because $\sum_{\mathfrak{a}|\mathfrak{d}} \mu(\mathfrak{a}) = 0$ if $\mathfrak{d} \neq \mathfrak{e}$. Hence $\sum_{\mathfrak{a}|\mathfrak{d}} \mu(\mathfrak{a})\theta_{\mathfrak{n}}(\mathfrak{a}\mathfrak{c}) \in R$. This completes the proof. □

For an ideal $\mathfrak{d}$ of $\mathbb{A}$, we write $\delta_{\mathfrak{n},\mathfrak{d}} := \delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{e})$ for simplicity.

Lemma 2.3. *For any prime ideal* $\mathcal{L}$ *of* $\mathcal{O}_{\mathcal{K}}$ *with* $\mathcal{L} \nmid \mathfrak{p}\mathfrak{n}$*, we have*

$$\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{p}}} = (x) \textit{ with } x \equiv 1 \bmod \mathfrak{p}.$$

Proof. Following the idea of Hayes in the end of [Ha, §2], we may assume that $\mathcal{L}$ splits completely in $\mathcal{K}$. Take the place $\mathfrak{l}$ under $\mathcal{L}$ as the infinite place ∞' of k. Now let ϕ be a sgn-normalized rank one Drinfeld module on $\mathbb{A}_{\infty'}$, which is the ring of functions in k regular away from ∞'. Let $\mathfrak{n}', \mathfrak{p}'$ and $\mathfrak{f}'$ be the ideals of $\mathbb{A}_{\infty'}$ associated to $\mathfrak{n}, \mathfrak{p}$ and $\mathfrak{f}$, respectively. Let $\mathcal{H}$ be the maximal real subfield of the cyclotomic function field of $(k, \infty', \mathrm{sgn})$ of conductor $\mathfrak{n}'$. Then $\mathcal{K}$ is contained in $\mathcal{H}$, and we proceed inside $\mathcal{H}$, as was done in [Ha, §6]. It is shown by Hayes in [Ha] that $\mathcal{L}^{\theta_{\mathfrak{n}}} = (\lambda_{\mathfrak{n}'})$, for some properly chosen primitive $\mathfrak{n}'$-torsion point $\lambda_{\mathfrak{n}'}$ of ϕ. If $\mathfrak{p} \nmid \mathfrak{n}$, then $\delta_{\mathfrak{n},\mathfrak{p}} = (N(\mathfrak{p}) - \sigma_{\mathfrak{p}})\theta_{\mathfrak{n}}$. Thus we have

$$\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{p}}} = (\lambda_{\mathfrak{n}'}^{N(\mathfrak{p})-\sigma_{\mathfrak{p}}}) \text{ with } \lambda_{\mathfrak{n}'}^{N(\mathfrak{p})-\sigma_{\mathfrak{p}}} \equiv 1 \bmod \mathfrak{p},$$

since $\mathfrak{p}$ is unramified in $\mathcal{K}$.

Now we assume that $\mathfrak{p}|\mathfrak{n}$, and let $\mathfrak{f} = \mathfrak{n}\mathfrak{p}^{-1}$ and $H = \mathrm{Gal}(\mathcal{K}/K_{\mathfrak{f}})$. In this case, by Lemma 2.1 (ii), we have

$$\delta_{\mathfrak{n},\mathfrak{p}} = N(\mathfrak{p})\theta_{\mathfrak{n}} - \theta'_{\mathfrak{f}} = \begin{cases} N(\mathfrak{p})\theta_{\mathfrak{n}} - s(H)\theta_{\mathfrak{n}}, & \text{if } \mathfrak{p}|\mathfrak{f}, \\ N(\mathfrak{p})\theta_{\mathfrak{n}} - s(H)\theta_{\mathfrak{n}} - \mathrm{Cor}_{\mathcal{K}/K_{\mathfrak{f}}}(\sigma_{\mathfrak{p}}^{-1}\theta_{\mathfrak{f}}), & \text{if } \mathfrak{p} \nmid \mathfrak{f}. \end{cases}$$

If $\mathfrak{p}|\mathfrak{f}$, then $\lambda_{\mathfrak{n}'}^{s(H)} = \phi_{\mathfrak{p}'}(\lambda_{\mathfrak{n}'}) \equiv \lambda_{\mathfrak{n}'}^{N(\mathfrak{p})} \bmod \mathfrak{p}'$. Thus

$$\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{p}}} = (\lambda_{\mathfrak{n}'}^{N(\mathfrak{p})-s(H)}) \text{ with } \lambda_{\mathfrak{n}'}^{N(\mathfrak{p})-s(H)} \equiv 1 \bmod \mathfrak{p}'.$$

If $\mathfrak{p} \nmid \mathfrak{f}$, then, for any $\sigma \in H$, σ acts on $\lambda_{\mathfrak{n}'}$ as ϕ_a for some $a \in (\mathbb{A}_{\infty'}/\mathfrak{n}')^*$ with $a \equiv 1 \bmod \mathfrak{f}'$. Also there is a unique $b \in \mathbb{A}_{\infty'}/\mathfrak{n}'$ with $b \equiv 1 \bmod \mathfrak{f}'$ but $b \equiv 0 \bmod \mathfrak{p}'$. Write $(b) = \mathfrak{p}'\mathfrak{r}'$. Then $\phi_b(\lambda_{\mathfrak{n}'}) = \phi_{\mathfrak{r}'}(\lambda_{\mathfrak{f}'}) = \lambda_{\mathfrak{f}'}^{\sigma_{\mathfrak{p}'}^{-1}}$. It is easy to see that

$$\prod_{\substack{a \in \mathbb{A}_{\infty'}/\mathfrak{n}' \\ a \equiv 1 \bmod \mathfrak{f}'}} \phi_a(\lambda_{\mathfrak{n}'}) = \phi_{\mathfrak{p}'}(\lambda_{\mathfrak{n}'}).$$

Thus

$$\lambda_{\mathfrak{n}'}^{s(H)} = \phi_{\mathfrak{p}'}(\lambda_{\mathfrak{n}'})/\lambda_{\mathfrak{f}'}^{\sigma_{\mathfrak{p}'}^{-1}}.$$

As before

$$\phi_{\mathfrak{p}'}(\lambda_{\mathfrak{n}'}) \equiv \lambda_{\mathfrak{n}'}^{N(\mathfrak{p})} \bmod \mathfrak{p}'.$$

Since $\mathcal{L}^{\mathrm{Cor}_{\mathcal{K}/K_{\mathfrak{f}}}(\sigma_{\mathfrak{p}}^{-1}\theta_{\mathfrak{f}})} = (\lambda_{\mathfrak{f}'}^{\sigma_{\mathfrak{p}'}^{-1}})$, we have

$$\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{p}}} = \big(\lambda_{\mathfrak{n}'}^{N(\mathfrak{p})}/\phi_{\mathfrak{p}'}(\lambda_{\mathfrak{n}'})\big) \text{ with } \lambda_{\mathfrak{n}'}^{N(\mathfrak{p})}/\phi_{\mathfrak{p}'}(\lambda_{\mathfrak{n}'}) \equiv 1 \bmod \mathfrak{p}'.$$

This completes the proof. □

Lemma 2.4. $\delta_{\mathfrak{n},\mathfrak{p}}(\mathfrak{c}) = \big(\mathrm{Cor}_{\mathcal{K}/K_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})}}(\delta_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c}),\mathfrak{p}})\big)^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}}$.

Proof. Note first that

$$\delta_{\mathfrak{n},\mathfrak{p}}(\mathfrak{c}) = N(\mathfrak{p})(\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})})^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}} - (\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{p}\mathfrak{c})})^{\sigma_{\mathfrak{p}\mathfrak{c}/(\mathfrak{n},\mathfrak{p}\mathfrak{c})}}. \tag{2.1}$$

Case 1 $\mathfrak{p} \nmid \mathfrak{n}$: In this case $(\mathfrak{n}, \mathfrak{p}\mathfrak{c}) = (\mathfrak{n}, \mathfrak{c})$, and so (2.1) becomes

$$(N(\mathfrak{p})\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})} - \theta'^{\sigma_{\mathfrak{p}}}_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})})^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}} = \big(\mathrm{Cor}_{\mathcal{K}/K_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})}}(\delta_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c}),\mathfrak{p}})\big)^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}}.$$

Case 2 $\mathfrak{p}|\mathfrak{n}$: In this case (2.1) becomes

$$N(\mathfrak{p})(\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})})^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}} - (\theta'_{\mathfrak{f}/(\mathfrak{f},\mathfrak{c})})^{\sigma_{\mathfrak{c}/(\mathfrak{f},\mathfrak{c})}}. \tag{2.2}$$

Write $\mathfrak{n} = \mathfrak{p}^i\mathfrak{f}'$ and $\mathfrak{c} = \mathfrak{p}^j\mathfrak{c}'$ with $(\mathfrak{p}, \mathfrak{f}'\mathfrak{c}') = \mathfrak{e}$. Then

$$(\mathfrak{n}, \mathfrak{c}) = \mathfrak{p}^{\min\{i,j\}}(\mathfrak{f}', \mathfrak{c}'), \qquad (\mathfrak{f}, \mathfrak{c}) = P^{\min\{i-1,j\}}(\mathfrak{f}', \mathfrak{c}'),$$
$$\frac{\mathfrak{n}}{(\mathfrak{n},\mathfrak{c})} = \mathfrak{p}^{i-\min\{i,j\}}\frac{\mathfrak{f}'}{(\mathfrak{f}',\mathfrak{c}')}, \qquad \frac{\mathfrak{c}}{(\mathfrak{n},\mathfrak{c})} = \mathfrak{p}^{j-\min\{i,j\}}\frac{\mathfrak{c}'}{(\mathfrak{f}',\mathfrak{c}')},$$
$$\frac{\mathfrak{f}}{(\mathfrak{f},\mathfrak{c})} = \mathfrak{p}^{i-1-\min\{i-1,j\}}\frac{\mathfrak{f}'}{(\mathfrak{f}',\mathfrak{c}')}, \qquad \frac{\mathfrak{c}}{(\mathfrak{f},\mathfrak{c})} = \mathfrak{p}^{j-\min\{i-1,j\}}\frac{\mathfrak{c}'}{(\mathfrak{f}',\mathfrak{c}')}.$$

If $j \geq i$, then $\mathfrak{f}/(\mathfrak{f},\mathfrak{c}) = \mathfrak{n}/(\mathfrak{n},\mathfrak{c})$ and $\mathfrak{c}/(\mathfrak{f},\mathfrak{c}) = \mathfrak{p}\mathfrak{c}/(\mathfrak{n},\mathfrak{c})$. Thus (2.2) becomes

$$\big(N(\mathfrak{p})\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})} - (\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})})^{\sigma_{\mathfrak{p}}}\big)^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}} = \big(\mathrm{Cor}_{\mathcal{K}/K_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})}}(\delta_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c}),\mathfrak{p}})\big)^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}}.$$

If $j < i$, then $\mathfrak{f}/(\mathfrak{f},\mathfrak{c}) = (\mathfrak{n},\mathfrak{c})/\mathfrak{p}$ and $\mathfrak{c}/(\mathfrak{f},\mathfrak{c}) = \mathfrak{c}/(\mathfrak{n},\mathfrak{c})$. Thus (2.2) becomes

$$\big(N(\mathfrak{p})\theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})} - \theta'_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})\mathfrak{p}}\big)^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}} = \big(\mathrm{Cor}_{\mathcal{K}/K_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c})}}(\delta_{\mathfrak{n}/(\mathfrak{n},\mathfrak{c}),\mathfrak{p}})\big)^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}}.$$

This completes the proof. □

Theorem 2.5. *For any $\mathfrak{d} \in T_0$, we have*

$$S_{\mathfrak{d}} \subseteq Ann_R(\mathcal{C}_{\mathfrak{d}}).$$

Proof. The case $\mathfrak{d} = \mathfrak{e}$ is proved by Tate-Deligne ([T]) and Hayes ([Ha]). Assume that $\mathfrak{d} \neq \mathfrak{e}$. It suffices to show that, for any prime ideal $\mathcal{L}$ of $\mathcal{O}_\mathcal{K}$ with $\mathcal{L} \nmid \mathfrak{d}\mathfrak{n}$, there exists an element $x \in \mathcal{K}$ such that $\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c})} = (x)$ with $x \equiv 1 \bmod \mathfrak{d}$.

Consider first the case $\mathfrak{d} = \mathfrak{p}^n$, a power of prime ideal $\mathfrak{p}$. For $\mathfrak{f}|\mathfrak{n}$, we have $\mathcal{L}^{\mathrm{Cor}_{\mathcal{K}/K_\mathfrak{f}}(\theta)} = N_{\mathcal{K}/K_\mathfrak{f}}(\mathcal{L})^\theta$ for any $\theta \in \mathbb{Z}[G_\mathfrak{f}]$. Thus, by Lemma 2.3 and Lemma 2.4, there exists $y \in \mathcal{K}$ such that

$$\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{p}}(\mathfrak{c})} = (y) \text{ with } y \equiv 1 \bmod \mathfrak{p}. \tag{2.3}$$

Raising (2.3) to the $N(\mathfrak{p}^{n-1})$-power, we find an element $x \in \mathcal{K}$ such that

$$\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{p}^n}(\mathfrak{c})} = (x) \text{ with } x \equiv 1 \bmod \mathfrak{p}^n.$$

Next we assume that $\mathfrak{d}$ has at least two distinct prime divisors. Since $\mu(\mathfrak{a}) = 0$ for any $\mathfrak{a}|\mathfrak{d}$ with $\mathfrak{a} \nmid \bar{\mathfrak{d}}$, we have $\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c}) = \frac{N(\mathfrak{d})}{N(\bar{\mathfrak{d}})}\delta_{\mathfrak{n},\bar{\mathfrak{d}}}(\mathfrak{c})$. For any prime ideal $\mathfrak{p}|\mathfrak{d}$, we have

$$\begin{aligned}
\mathcal{L}^{\delta_{\mathfrak{n},\bar{\mathfrak{d}}}(\mathfrak{c})} &= \prod_{\mathfrak{a}|\bar{\mathfrak{d}}/\mathfrak{p}} (\mathcal{L}^{\theta_\mathfrak{n}(\mathfrak{a}\mathfrak{c})})^{\mu(\mathfrak{a})\frac{N(\bar{\mathfrak{d}})}{N(\mathfrak{a})}} \times \prod_{\mathfrak{a}|\bar{\mathfrak{d}}/\mathfrak{p}} (\mathcal{L}^{\theta_\mathfrak{n}(\mathfrak{p}\mathfrak{a}\mathfrak{c})})^{\mu(\mathfrak{p}\mathfrak{a})\frac{N(\bar{\mathfrak{d}})}{N(\mathfrak{p}\mathfrak{a})}} \\
&= \prod_{\mathfrak{a}|\bar{\mathfrak{d}}/\mathfrak{p}} (\mathcal{L}^{\theta_\mathfrak{n}(\mathfrak{a}\mathfrak{c})N(\mathfrak{p})-\theta_\mathfrak{n}(\mathfrak{p}\mathfrak{a}\mathfrak{c})})^{\mu(\mathfrak{a})\frac{N(\bar{\mathfrak{d}})}{N(\mathfrak{p}\mathfrak{a})}} \\
&= \prod_{\mathfrak{a}|\bar{\mathfrak{d}}/\mathfrak{p}} (\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{p}}(\mathfrak{a}\mathfrak{c})})^{\mu(\mathfrak{a})\frac{N(\bar{\mathfrak{d}})}{N(\mathfrak{p}\mathfrak{a})}} \\
&= \prod_{\mathfrak{a}|\bar{\mathfrak{d}}/\mathfrak{p}} (x_\mathfrak{a})^{\mu(\mathfrak{a})\frac{N(\bar{\mathfrak{d}})}{N(\mathfrak{p}\mathfrak{a})}}, \text{where } \mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{p}}(\mathfrak{a}\mathfrak{c})} = (x_\mathfrak{a}) \text{ with } x_\mathfrak{a} \equiv 1 \bmod \mathfrak{p} \\
&= (x_0), \text{ where } x_0 = \prod_{\mathfrak{a}|\bar{\mathfrak{d}}/\mathfrak{p}} (x_\mathfrak{a})^{\mu(\mathfrak{a})\frac{N(\bar{\mathfrak{d}})}{N(\mathfrak{p}\mathfrak{a})}} \equiv 1 \bmod \mathfrak{p}.
\end{aligned}$$

Thus

$$\mathcal{L}^{\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c})} = (x) \text{ with } x = (x_0)^{\frac{N(\mathfrak{d})}{N(\bar{\mathfrak{d}})}} \equiv 1 \bmod \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(\mathfrak{d})}$$

for any prime ideal $\mathfrak{p}|\mathfrak{d}$. This completes the proof. □

3. The minus part of the ray class groups

Let $\mathcal{C}_\mathfrak{d}(\mathcal{K}^+)$ denote the $\mathfrak{d}$-ray class group of $\mathcal{K}^+$ and $j_\mathfrak{d} : \mathcal{C}_\mathfrak{d}(\mathcal{K}^+) \to \mathcal{C}_\mathfrak{d}$ be the map induced by the inclusion map on ideals from $\mathcal{K}^+$ to $\mathcal{K}$. Let $N^{(\mathfrak{d})}_{\mathcal{K}/\mathcal{K}^+} : \mathcal{C}_\mathfrak{d} \to \mathcal{C}_\mathfrak{d}(\mathcal{K}^+)$ be the norm map.

Lemma 3.1. (i) *If $\mathfrak{d} \neq \mathfrak{e}$, then $j_\mathfrak{d}$ is injective.*
(ii) *The cokernel of $N^{(\mathfrak{d})}_{\mathcal{K}/\mathcal{K}^+}$ has exponent $q-1$, i.e.,*

$$\mathcal{C}_\mathfrak{d}(\mathcal{K}^+)^{q-1} \subseteq N^{(\mathfrak{d})}_{\mathcal{K}/\mathcal{K}^+}(\mathcal{C}_\mathfrak{d}) \subseteq \mathcal{C}_\mathfrak{d}(\mathcal{K}^+).$$

Proof. (i) Let $\mathfrak{A}$ be an ideal of $\mathcal{K}^+$ and assume $\mathfrak{A} = (z)$ with $z \in \mathcal{K}$ and $z \equiv 1 \bmod \mathfrak{d}$. Then $(z^j) = (z)$, where j is a generator of J. Thus $z^{1-j} \in \mathcal{O}^*_\mathcal{K}$. For any infinite prime $\mathfrak{P}_\infty$ of $\mathcal{K}$, $|z^{1-j}|_{\mathfrak{P}_\infty} = 1$. Thus $z^{1-j} \in \mathbb{F}^*_q$ with $z^{1-j} \equiv 1 \bmod \mathfrak{d}$. Since $\mathfrak{d} \neq \mathfrak{e}$, $z^{1-j} = 1$ and so $z \in \mathcal{K}^+$. Thus $\mathfrak{A} = (z)$ in $\mathcal{K}^+$. Hence $j_\mathfrak{d}$ is injective.

(ii) For any $\mathfrak{C} \in \mathcal{C}_{\mathfrak{d}}(\mathcal{K}^+)$, we have

$$\mathfrak{C}^{q-1} = \mathfrak{C}^{1+j+\cdots+j^{q-2}} = N^{(\mathfrak{d})}_{\mathcal{K}/\mathcal{K}^+}(\mathfrak{C}).$$

Thus we get the result. □

Let $\mathcal{O}^*_{\mathcal{K},\mathfrak{d}} = \{x \in \mathcal{O}^*_{\mathcal{K}} : x \equiv 1 \bmod \mathfrak{d}\}$ and $\mathcal{O}^*_{\mathcal{K}^+,\mathfrak{d}} = \mathcal{O}^*_{\mathcal{K}^+} \cap \mathcal{O}^*_{\mathcal{K},\mathfrak{d}}$.

Lemma 3.2. *If $\mathfrak{d} \neq \mathfrak{e}$, then $\mathcal{O}^*_{\mathcal{K},\mathfrak{d}} = \mathcal{O}^*_{\mathcal{K}^+,\mathfrak{d}}$.*

Proof. For any $x \in \mathcal{O}^*_{\mathcal{K},\mathfrak{d}}$, as in the proof of Proposition 1.1 in [Hr], we have $x^{1-j} \in \mathbb{F}_q^*$. But $x^{1-j} \equiv 1 \bmod \mathfrak{d}$, so $x^{1-j} = 1$. Thus $x \in \mathcal{O}^*_{\mathcal{K}^+}$. Hence $\mathcal{O}^*_{\mathcal{K},\mathfrak{d}} = \mathcal{O}^*_{\mathcal{K}^+,\mathfrak{d}}$. □

Let $\widehat{G}$ be the group of characters of G with values in $\mathbb{C}^*$. A character χ is called *real* if $\chi(J) = 1$, and *non-real* otherwise. Let $\widehat{G}^-$ denote the set of all non-real characters of G. The conductor $\mathfrak{f}_\chi$ of a character χ is the smallest integral ideal $\mathfrak{m}$ such that χ factors through $G_{\mathfrak{m}}$. We denote by χ_1 the trivial character. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{A}$. We define $\chi(\mathfrak{p})$ as follows. If $\mathfrak{p} \nmid \mathfrak{f}_\chi$, let $\sigma_{\mathfrak{p}}$ is the Artin automorphism associated to $\mathfrak{p}$ in $G_{\mathfrak{f}_\chi}$ and let $\chi(\mathfrak{p}) = \chi(\sigma_{\mathfrak{p}})$. If $\mathfrak{p} \mid \mathfrak{f}_\chi$, we put $\chi(\mathfrak{p}) = 0$.

Recall that $\mathcal{C}^-_{\mathfrak{d}} = \{\mathfrak{c} \in \mathcal{C}_{\mathfrak{d}} : s(J) \cdot \mathfrak{c} = 0\}$, which is also the kernel of $N^{(\mathfrak{d})}_{\mathcal{K}/\mathcal{K}^+}$. Set $h^-_{\mathfrak{d}} := |\mathcal{C}^-_{\mathfrak{d}}|$, called the minus $\mathfrak{d}$-ray class number of $\mathcal{K}$.

Theorem 3.3. *If $\mathfrak{d} \neq \mathfrak{e}$, then*

$$h^-_{\mathfrak{d}} = h^-_{\mathfrak{e}} \left(N(\mathfrak{d})^{\frac{q-2}{q-1}h\phi(\mathfrak{n})} \varrho_{\mathcal{K},\mathfrak{d}}/Q_0\right) \prod_{\mathfrak{p}|\mathfrak{d}} \prod_{\chi \in \widehat{G}^-} (1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-1}),$$

where $Q_0 = (\mathcal{O}^*_{\mathcal{K}} : \mathcal{O}^*_{\mathcal{K}^+})$, $\varrho_{\mathcal{K},\mathfrak{d}} = |\operatorname{Coker}(N^{(\mathfrak{d})}_{\mathcal{K}/\mathcal{K}^+})|$ and $\mathfrak{p}$ runs over all prime ideals of $\mathbb{A}$ dividing $\mathfrak{d}$.

Proof. Following the arguments in [L, Chap VI, §1] with Lemma 3.2, we have

$$\frac{|\mathcal{C}_{\mathfrak{d}}|}{|\mathcal{C}_{\mathfrak{d}}(\mathcal{K}^+)|} = h^-_{\mathfrak{e}} \frac{|(\mathcal{O}_{\mathcal{K}}/\mathfrak{d}\mathcal{O}_{\mathcal{K}})^*|}{|(\mathcal{O}_{\mathcal{K}^+}/\mathfrak{d}\mathcal{O}_{\mathcal{K}^+})^*|} \frac{1}{Q_0}.$$

Thus it follows from the exact sequence

$$1 \longrightarrow \mathcal{C}^-_{\mathfrak{d}} \longrightarrow \mathcal{C}_{\mathfrak{d}} \xrightarrow{N^{(\mathfrak{d})}_{\mathcal{K}/\mathcal{K}^+}} \mathcal{C}_{\mathfrak{d}}(\mathcal{K}^+) \longrightarrow \operatorname{Coker}(N^{(\mathfrak{d})}_{\mathcal{K}/\mathcal{K}^+}) \longrightarrow 1,$$

that

$$h^-_{\mathfrak{d}} = h^-_{\mathfrak{e}} \frac{|(\mathcal{O}_{\mathcal{K}}/\mathfrak{d}\mathcal{O}_{\mathcal{K}})^*|}{|(\mathcal{O}_{\mathcal{K}^+}/\mathfrak{d}\mathcal{O}_{\mathcal{K}^+})^*|} \frac{\varrho_{\mathcal{K},\mathfrak{d}}}{Q_0}.$$

Now, the result follows from the following facts that

$$\frac{|(\mathcal{O}_{\mathcal{K}}/\mathfrak{d}\mathcal{O}_{\mathcal{K}})^*|}{|(\mathcal{O}_{\mathcal{K}^+}/\mathfrak{d}\mathcal{O}_{\mathcal{K}^+})^*|} = N(\mathfrak{d})^{\frac{q-2}{q-1}h\phi(\mathfrak{n})} \frac{\prod_{\mathfrak{P}}(1 - N(\mathfrak{P})^{-1})}{\prod_{\mathfrak{P}^+}(1 - N(\mathfrak{P}^+)^{-1})}$$

and

$$\frac{\prod_{\mathfrak{P}}(1 - N(\mathfrak{P})^{-1})}{\prod_{\mathfrak{P}^+}(1 - N(\mathfrak{P}^+)^{-1})} = \prod_{\mathfrak{p}|\mathfrak{d}} \prod_{\chi \in \widehat{G}^-} (1 - \chi(\mathfrak{p}) N(\mathfrak{p})^{-1}),$$

where $\mathfrak{P}$ (resp. $\mathfrak{P}^+$) runs over all prime ideals of $\mathcal{O}_{\mathcal{K}}$ (resp. $\mathcal{O}_{\mathcal{K}^+}$) dividing $\mathfrak{d}$ and $\mathfrak{p}$ runs over all prime ideals of $\mathbb{A}$ dividing $\mathfrak{d}$. □

4. ℓ-PART OF THE INDEX $(R^- : S_{\mathfrak{d}}^-)$

For a prime ideal $\mathfrak{p}$ of $\mathbb{A}$, let $T_{\mathfrak{p}}$ be the inertia group of $\mathfrak{p}$ in G and let $F_{\mathfrak{p}} \in G$ be any Frobenius automorphism for $\mathfrak{p}$, which is well defined modulo $T_{\mathfrak{p}}$. In $\mathbb{Q}[G]$, we define

$$\overline{\sigma}_{\mathfrak{p}} := F_{\mathfrak{p}}^{-1} \cdot \frac{s(T_{\mathfrak{p}})}{|T_{\mathfrak{p}}|}$$

and $\mathcal{U}_{\mathfrak{p}} := R \cdot s(T_{\mathfrak{p}}) + R \cdot (1 - \overline{\sigma}_{\mathfrak{p}})$. We also define $\mathcal{U}_{\mathfrak{s}} := \prod_{\mathfrak{p}|\mathfrak{s}} \mathcal{U}_{\mathfrak{p}}$ at any $\mathfrak{s}|\bar{\mathfrak{n}}$.

Lemma 4.1. *For any $\mathfrak{s}|\bar{\mathfrak{n}}$, the index $(\varepsilon^- R : \varepsilon^- \mathcal{U}_{\mathfrak{s}})$ is a power of $q-1$.*

Proof. It suffices to show that $(\varepsilon^- \mathcal{U}_{\mathfrak{s}} : \varepsilon^- \mathcal{U}_{\mathfrak{s}\mathfrak{p}})$ is a power of $q-1$ for $\mathfrak{s}\mathfrak{p}|\bar{\mathfrak{n}}$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{A}$. Since the multiplication by $1-j$ on $\mathbb{Q}[G]^-$ is injective, by Lemma 6.1 in [Si], we have

$$(\varepsilon^- \mathcal{U}_{\mathfrak{s}} : \varepsilon^- \mathcal{U}_{\mathfrak{s}\mathfrak{p}}) = ((1-j)\mathcal{U}_{\mathfrak{s}} : (1-j)\mathcal{U}_{\mathfrak{s}\mathfrak{p}}),$$

which is a power of $q-1$ ([Y1, §6]). □

Let e_χ be the idempotent element associated $\chi \in \widehat{G}$. Set

$$\omega := \sum_{\chi_1 \neq \chi \in \widehat{G}} L(0, \overline{\chi}) e_\chi,$$

where $L(s, \chi)$ is the Artin L-function attached to χ. For $\mathfrak{f}|\mathfrak{n}$, let $I_{\mathfrak{f}} = \mathrm{Gal}(\mathcal{K}/K_{\mathfrak{f}})$. We also let

$$\alpha_{\mathfrak{f}} := s(I_{\mathfrak{f}}) \prod_{\mathfrak{p}|\mathfrak{f}} (1 - \overline{\sigma}_{\mathfrak{p}}) \quad \text{if } \mathfrak{f} \neq \mathfrak{e}$$

and $\alpha_{\mathfrak{e}} := s(I_{\mathfrak{e}})$. Then we have

Lemma 4.2. *For any $\mathfrak{f}|\mathfrak{n}$, $\varepsilon^- \theta_{\mathfrak{n}}(\mathfrak{f}) = \varepsilon^- \omega \alpha_{\mathfrak{n}\mathfrak{f}^{-1}}$.*

Proof. See the proof of Lemma 6 in [Y3]. □

In the following we assume that $\mathfrak{d} \neq \mathfrak{e}$ and $\bar{\mathfrak{d}}|\mathfrak{n}$.

Lemma 4.3. *$S_{\mathfrak{d}}$ is generated as an R-module by $\{\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c}) : \mathfrak{c}|\mathfrak{n}\}$.*

Proof. Since $\mathfrak{d} \neq \mathfrak{e}$, $S_{\mathfrak{d}}$ is generated as an R-module by $\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c})$ for all $\mathfrak{c}$ mod $\sim_{\mathfrak{n}}$ by Proposition 2.2. Since $\theta_{\mathfrak{n}}(\mathfrak{c}) = \theta_{\mathfrak{n}}((\mathfrak{n}, \mathfrak{c}))^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}}$, we have

$$\delta_{\mathfrak{n},\mathfrak{d}}(\mathfrak{c}) = \delta_{\mathfrak{n},\mathfrak{d}}((\mathfrak{n}, \mathfrak{c}))^{\sigma_{\mathfrak{c}/(\mathfrak{n},\mathfrak{c})}}.$$

This completes the proof. □

For $\mathfrak{s} \in T_0$, we write

$$\mathfrak{n}_{\mathfrak{s}} := \prod_{\mathfrak{p}|\mathfrak{s}} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})}.$$

Let $\mathfrak{d}_1 = \prod_{\mathfrak{p}|\mathfrak{d}} \mathfrak{p}^{-\mu(\mathfrak{n}_{\mathfrak{p}})}$. For $\mathfrak{p}|\bar{\mathfrak{d}}/\mathfrak{d}_1$ let $\mathcal{B}_{\mathfrak{p}}$ be the R-module generated by the elements

$$\eta_{\mathfrak{p}} := N(\mathfrak{p})s(I_{\mathfrak{p}\mathfrak{n}/\mathfrak{n}_{\mathfrak{p}}})(1-\overline{\sigma}_{\mathfrak{p}}) - s(T_{\mathfrak{p}}) \text{ and } \gamma_{\mathfrak{p},\mathfrak{p}^i} := N(\mathfrak{p})s(I_{\mathfrak{n}/\mathfrak{p}^i}) - s(I_{\mathfrak{n}/\mathfrak{p}^{i+1}})$$

for $0 \le i \le \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) - 2$, and for $\mathfrak{p}|\mathfrak{d}_1$ we set $\mathcal{B}_{\mathfrak{p}} := R \cdot \eta_{\mathfrak{p}}$.

Using Lemma 4.2 and Lemma 4.3, we follow exactly the same process as in the classical case ([Sc, §4.2]) to get the following proposition. We remark that S_d (resp. $\mathfrak{d}_d(x)$) in [Sc, Lemma 4.2.2] should be replaced by $\varepsilon^- S_d$ (resp. $\varepsilon^-\mathfrak{d}_d(x)$).

Proposition 4.4. $\varepsilon^- S_{\mathfrak{d}} = \mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}}} \cdot \prod_{\mathfrak{p}|\bar{\mathfrak{d}}} \mathcal{B}_{\mathfrak{p}} \cdot \varepsilon^- \omega \frac{N(\mathfrak{d})}{N(\bar{\mathfrak{d}})}$.

Let ℓ be a prime number. Let $R_\ell = \mathbb{Z}_\ell[G], S_{\mathfrak{d},\ell} = S_{\mathfrak{d}} \otimes \mathbb{Z}_\ell$ and $\mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}},\ell} = \mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}}} \otimes \mathbb{Z}_\ell$. Note that if $\ell \neq p$, then $S_{\mathfrak{d},\ell} = S_{\bar{\mathfrak{d}},\ell}$. For any prime ideal $\mathfrak{p}|\mathfrak{d}$, set

$$\kappa_{\mathfrak{p}} := s(I_{\mathfrak{p}\mathfrak{n}/\mathfrak{n}_{\mathfrak{p}}})(1 - N(\mathfrak{p})(1-\overline{\sigma}_{\mathfrak{p}})) + s(T_{\mathfrak{p}}) - N(\mathfrak{n}_{\mathfrak{p}}/\mathfrak{p}).$$

Then

$$\kappa_{\mathfrak{p}} = (s(I_{\mathfrak{p}\mathfrak{n}/\mathfrak{n}_{\mathfrak{p}}}) - N(\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})-1}) - \eta_{\mathfrak{p}}.$$

Especially, if $\mathfrak{p}|\mathfrak{d}_1$, then $\kappa_{\mathfrak{p}} = -\eta_{\mathfrak{p}}$, and so $\mathcal{B}_{\mathfrak{p}} = R \cdot \kappa_{\mathfrak{p}}$. For $\mathfrak{p}|\bar{\mathfrak{d}}/\mathfrak{d}_1$, it follows from the definition of $\gamma_{\mathfrak{p},\mathfrak{p}^i}$ that

$$s(I_{\mathfrak{p}\mathfrak{n}/\mathfrak{n}_{\mathfrak{p}}}) = N(\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})-1} - \sum_{j=0}^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{n})-2} N(\mathfrak{p})^{\mathrm{ord}_{\mathfrak{p}}(\mathfrak{p})-2-j}\gamma_{\mathfrak{p},\mathfrak{p}^j}.$$

Thus $\kappa_{\mathfrak{p}} \in \mathcal{B}_{\mathfrak{p}}$, and so $R \cdot \kappa_{\mathfrak{p}} \subseteq \mathcal{B}_{\mathfrak{p}}$. Set

$$\kappa := \prod_{\mathfrak{p}|\bar{\mathfrak{d}}} \kappa_{\mathfrak{p}},$$

and $\mathcal{B}_{\mathfrak{p},\ell} := \mathcal{B}_{\mathfrak{p}} \otimes \mathbb{Z}_\ell$ for any prime ideal $\mathfrak{p}|\mathfrak{d}$.

Proposition 4.5. *Let ℓ be a prime number with $\ell \neq p$. Then $\mathcal{B}_{\mathfrak{p},\ell} = R_\ell \cdot \kappa_{\mathfrak{p}}$ for any prime ideal $\mathfrak{p}|\mathfrak{d}$, hence*

$$\varepsilon^- S_{\mathfrak{d},\ell} = \mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}},\ell} \cdot \varepsilon^- \kappa\omega.$$

Proof. We only need to consider the case $v = \mathrm{ord}_{\mathfrak{p}}(\mathfrak{n}) \ge 2$. Set

$$\epsilon_i := N(\mathfrak{p})^{-i} s(I_{\mathfrak{n}/\mathfrak{p}^i}) \in \mathbb{Z}_\ell[G]$$

for $0 \le i < v$. It is easy to see that $\epsilon_{v-1} \cdot \kappa_{\mathfrak{p}} = -\eta_{\mathfrak{p}}$, from where we have $\eta_{\mathfrak{p}} \in R_\ell \cdot \kappa_{\mathfrak{p}}$. We also have

$$N(\mathfrak{p})^{i+1-v}(1-\epsilon_i)\kappa_{\mathfrak{p}} = s(I_{\mathfrak{n}/\mathfrak{p}^i}) - N(\mathfrak{p})^i.$$

Thus we have

$$\begin{aligned} \gamma_{\mathfrak{p},\mathfrak{p}^i} &= -\big(s(I_{\mathfrak{n}/\mathfrak{p}^{i+1}}) - N(\mathfrak{p})^{i+1}\big) + N(\mathfrak{p})\big(s(I_{\mathfrak{n}/\mathfrak{p}^i}) - N(\mathfrak{p})^i\big) \\ &= N(\mathfrak{p})^{i+2-v}(\epsilon_i - \epsilon_{i+1})\kappa_{\mathfrak{p}} \in R_\ell \cdot \kappa_{\mathfrak{p}}. \end{aligned}$$

This completes the proof. $\square$

Lemma 4.6. *For any prime ideal $\mathfrak{p}|\mathfrak{n}$ and a character $\chi \in \widehat{G}$, we have*

$$|\chi(\kappa_{\mathfrak{p}})|_\ell = |1-\chi(\mathfrak{p})N(\mathfrak{p})^{-1}|_\ell \text{ if } \ell \neq p$$

and

$$|\chi(\kappa_{\mathfrak{p}})|_p = \begin{cases} N(\mathfrak{n}_{\mathfrak{p}})^{-1}|1-\chi(\mathfrak{p})N(\mathfrak{p})^{-1}|_p & \text{if } \chi \text{ is trivial on } I_{\mathfrak{pn}/\mathfrak{n}_{\mathfrak{p}}}, \\ N(\mathfrak{n}_{\mathfrak{p}}/\mathfrak{p})^{-1}|1-\chi(\mathfrak{p})N(\mathfrak{p})^{-1}|_p & \text{otherwise.} \end{cases}$$

Proof. If $\mathfrak{p}|\mathfrak{f}_\chi$, then $\chi(\mathfrak{p}) = 0$ and χ is non-trivial on $T_{\mathfrak{p}}$. Thus $\chi(s(T_{\mathfrak{p}})) = 0$, and so

$$\chi(\kappa_{\mathfrak{p}}) = \chi(s(I_{\mathfrak{pn}/\mathfrak{n}_{\mathfrak{p}}}))(1-N(\mathfrak{p})) - N(\mathfrak{n}_{\mathfrak{p}}/\mathfrak{p}),$$

which is equal to $-N(\mathfrak{n}_{\mathfrak{p}})$ or $-N(\mathfrak{n}_{\mathfrak{p}}/\mathfrak{p})$ according as χ is trivial or not on $I_{\mathfrak{pn}/\mathfrak{n}_{\mathfrak{p}}}$.

If $\mathfrak{p} \nmid \mathfrak{f}_\chi$, then χ is trivial on $T_{\mathfrak{p}}$ (especially on $I_{\mathfrak{pn}/\mathfrak{n}_{\mathfrak{p}}}$), and so

$$\chi(\kappa_{\mathfrak{p}}) = N(\mathfrak{n}_{\mathfrak{p}}/\mathfrak{p})(N(\mathfrak{p})\chi(\mathfrak{p})-1) = N(\mathfrak{n}_{\mathfrak{p}})\chi(\mathfrak{p})^{-1}(1-\chi(\mathfrak{p})N(\mathfrak{p})^{-1}).$$

This completes the proof. $\square$

Theorem 4.7. *Let ℓ be a prime number with $\ell \nmid p(q-1)$. For any $\mathfrak{d} \in T_0^*$ with $\bar{\mathfrak{d}}|\mathfrak{n}$, the ℓ-part of $(R^- : S_{\mathfrak{d}}^-)$ is equal to the ℓ-part of $|\mathcal{C}_{\mathfrak{d}}^-|$.*

Proof. Note that the ℓ-part of $(R^- : S_{\mathfrak{d}}^-)$ is equal to $(R_\ell^- : S_{\mathfrak{d},\ell}^-)$. Thus it suffices to show that $(R_\ell^- : S_{\mathfrak{d},\ell}^-)$ is equal to the ℓ-part of $|\mathcal{C}_{\mathfrak{d}}^-|$. By the equation (a) in [Y3], Lemma 4.1 and the fact that $(q-1)\varepsilon^- S_{\mathfrak{d},\ell} \subseteq S_{\mathfrak{d},\ell}^-$, we have

$$(R_\ell^- : \varepsilon^- R_\ell) = (\varepsilon^- R_\ell : \varepsilon^- \mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}},\ell}) = (\varepsilon^- S_{\mathfrak{d},\ell} : S_{\mathfrak{d},\ell}^-) = 1.$$

Thus $(R_\ell^- : S_{\mathfrak{d},\ell}^-) = (\varepsilon^- \mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}},\ell} : \varepsilon^- S_{\mathfrak{d},\ell})$. Now following the same argument as [Sc, Theorem 3] using Theorem 3.3, Proposition 4.5 and Lemma 4.6, we get the result. $\square$

To consider the p-part of the index $(R^- : S_{\mathfrak{d}}^-)$, we have to compute the index $(\varepsilon^- \mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}},p} : \varepsilon^- \mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}},p} \prod_{\mathfrak{p}|\bar{\mathfrak{d}}} \mathcal{B}_{\mathfrak{p},p})$. It seems very difficult to compute this index because there may appear more than one $\mathcal{B}_{\mathfrak{p},p}$. Furthermore, the structure of $\mathcal{B}_{\mathfrak{p},p}$ is more complicated, since $I_{\mathfrak{pn}/\mathfrak{n}_{\mathfrak{p}}}$ is not cyclic. But when $\mathfrak{n}$ is square free so that $\mathfrak{d} = \mathfrak{d}_1$, then $\mathcal{B}_{\mathfrak{p},p} = R_p \cdot \kappa_{\mathfrak{p}}$ for any $\mathfrak{p}|\mathfrak{d}$, and so

$$\varepsilon^- S_{\mathfrak{d},p} = \mathcal{U}_{\bar{\mathfrak{n}}/\bar{\mathfrak{d}},p} \cdot \varepsilon^- \kappa\omega.$$

By Lemma 4.6, we have

$$|\chi(\kappa_{\mathfrak{p}})|_p = N(\mathfrak{p})^{-1}|1-\chi(\mathfrak{p})N(\mathfrak{p})^{-1}|_p,$$

and so the same process as in the proof of Theorem 4.7 gives

Theorem 4.8. *Assume that $\mathfrak{n}$ is square free. Then the p-part of $(R^- : S_{\mathfrak{d}}^-)$ is equal to the p-part of $|\mathcal{C}_{\mathfrak{d}}^-|$.*

Finally, we follow the same argument in the proof of Corollary 4.5.2 in [Sc] using Theorem 3.3, Theorem 4.7 and Theorem 4.8 to get

Corollary 4.9. *Let ℓ be a prime number with $\ell \nmid (q-1)$. Assume that $\mathfrak{n}$ is square free if $\ell = p$. For any $\mathfrak{d} \in T_0^*$ (not necessarily $\bar{\mathfrak{d}}|\mathfrak{n}$), the ℓ-part of $(R^- : S_{\mathfrak{d}}^-)$ is equal to the ℓ-part of $|\mathcal{C}_{\mathfrak{d}}^-|$.*

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Department of Mathematics, KAIST, Taejon 305-701, Korea
E-mail address: shbae@kaist.ac.kr

Department of Mathematics Education, Chungbuk National University, Cheongju 361-763, Korea
E-mail address: hyjung@chungbuk.ac.kr