A NOTE ON SUMS OF PRODUCTS OF BERNOULLI NUMBERS

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Abstract. In this paper we obtain new approach to closed expressions for sums of products of Bernoulli numbers by using the relation of values at non-positive integers of the important representation of the multiple Hurwitz zeta-function in terms of the Hurwitz zeta-function.

1. Introduction and preliminaries

Let n be a positive integer and $x \in \mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$.

Let $\zeta_n(s,x)$ denote the multiple Hurwitz zeta-function defined by

$$\zeta_n(s,x) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{1}{(x+k_1+\cdots+k_n)^s}, \quad \mathrm{Re}\,(s) > n \tag{1.1}$$

and for $\mathrm{Re}\,(s) \le n$; $s \ne 1,2,\ldots,n$ by their analytic continuations. Then, in terms of the familiar higher-order Bernoulli polynomials $B_k^{(n)}(x)$ defined by means of the generating function

$$\left(\frac{t}{e^t-1}\right)^n e^{xt} = \sum_{k=0}^{\infty} B_k^{(n)}(x)\frac{t^k}{k!}, \quad n \ge 1 \tag{1.2}$$

it is known that

$$\zeta_n(-k,x) = (-1)^n \frac{k!}{(k+n)!} B_{k+n}^{(n)}(x) \tag{1.3}$$

for $k = 0,1,\ldots.$ (cf. [2, 5, 14]). In particular we have $B_k^{(n)}(0) = B_k^{(n)}$, the higher-order Bernoulli numbers. When $n = 1$, $\zeta_1(s,x) = \zeta(x,s)$ is the well known Hurwitz zeta-function. The multiple Hurwitz zeta-function $\zeta_n(s,x)$ at positive integers which is greater than or equal to n is closely related the multiple gamma functions as a extension of the classical Euler gamma functions $\Gamma(x)$ (to be introduced below), the so-called Barnes multiple gamma functions $\Gamma_n(x)$ with the parameter x is defined by $\Gamma_n(x) = \exp(\frac{\partial}{\partial s}\zeta_n(s,x)|_{s=0}) = \prod_{k_1,\ldots,k_n=0}^{\infty}(x+k_1+\cdots+k_n)^{-1}$ (cf. [14]). The multiple gamma function, originally introduced over 100 years ago, has significant applications in the connection with the Riemann Hypothesis (cf. [14, 15]).

From (1.2), we easily obtain

$$B_k^{(n+1)} = \left(1-\frac{n}{k}\right)B_k^{(n)} - kB_{k-1}^{(n)}; \tag{1.4}$$

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$$B_k^{(n)} = -\frac{k}{n}\sum_{i=1}^{k}(-1)^i\binom{k}{i}B_iB_{k-i}^{(n)}; \tag{1.5}$$

$$B_k^{(n)}(x) = (-1)^k B_k^{(n)}(n-x); \tag{1.6}$$

$$B_k^{(n)}(x) = \sum_{\substack{k_1,\dots,k_n\geq 0\\ k_1+\cdots+k_n=k}}\binom{k}{k_1,\dots,k_n}B_{k_1}(x_1)\cdots B_{k_n}(x_n), \tag{1.7}$$

where $x = x_1 + \cdots + x_n$.

Set

$$x(x+1)\cdots(x+n-1) = \sum_{j=0}^{n}\begin{bmatrix} n\\ j\end{bmatrix} x^j, \tag{1.8}$$

where $\begin{bmatrix} n\\ j\end{bmatrix}$ are the Stirling cycle numbers, defined recursively by

$$\begin{bmatrix} n\\ j\end{bmatrix} = (n-1)\begin{bmatrix} n-1\\ j\end{bmatrix} + \begin{bmatrix} n-1\\ j-1\end{bmatrix}, \quad \begin{bmatrix} n\\ 0\end{bmatrix} = \begin{cases} 1, & n=0\\ 0, & n\neq 0.\end{cases}$$

Then

$$\binom{k+n-1}{n-1} = \sum_{l=0}^{n-1} p_{n,l}(x+n)(x+n+k)^l,$$

where $p_{n,l}(x+n)$ is a polynomial in x defined by (see, for details, [14])

$$p_{n,l}(x+n) = \frac{1}{(n-1)!}\sum_{j=l}^{n-1}(-1)^{j-l}(x+n)^{j-l}\binom{j}{l}\begin{bmatrix} n\\ j+1\end{bmatrix}. \tag{1.9}$$

By above equations, the multiple Hurwitz zeta-function defined by (1.1) may be expressed by means of the Hurwitz zeta-function

$$\zeta_n(s, x+n) = \sum_{l=0}^{n-1} p_{n,l}(x+n)\zeta(s-l, x+n), \quad x \geq 0, \tag{1.10}$$

which due to Mellin [8], Choi [2], Vardi [15] and Kanemitsu et al [5].

A well-known relation among the Bernoulli numbers is (for $n \geq 2$)

$$\sum_{i=1}^{k-1}\binom{2k}{2i}B_{2i}B_{2k-2i} = -(2k+1)B_{2k}, \quad k \geq 2. \tag{1.11}$$

This was found by many authors, including Euler; for references, see, e.g., [3, 6, 7, 9, 10, 11, 12, 13].

Eie [4] and Sitaramachandrarao and Davis [13] considered the sum of products of 3 and 4 Bernoulli numbers.

Dilcher [3] established the following interesting sums of product of Bernoulli numbers

$$\sum_{\substack{r_1,\dots,r_n\geq 0\\ r_1+\dots+r_n=r}} \binom{2r}{2r_1,\dots,2r_n} B_{2r_1}\cdots B_{2r_n}$$

$$(1.12)\qquad = \begin{cases} \dfrac{(2r)!}{(2r-n)!}\displaystyle\sum_{k=0}^{\left[\frac{n-1}{2}\right]} b_k^{(n)}\frac{B_{2r-2k}}{2r-2k}, & 2r>n\\ \dfrac{(2r)!}{4^r}+\displaystyle\sum_{k=0}^{r-1}\frac{(2r)!}{2r-2k}b_k^{(2r)}B_{2r-2k}, & 2r=n\\ (-1)^{n-1}(n-2r-1)!(2r)!b_r^{(n)}, & 0\leq r\leq\left[\dfrac{n-1}{2}\right], \end{cases}$$

where

$$\binom{2r}{2r_1,\dots,2r_n}=\frac{(2r)!}{(2r_1)!\cdots(2r_n)!}$$

is the multinomial coefficient and $b_k^{(n)}$ is the sequence of rational number defined by

$$b_0^{(1)}=1,\quad b_k^{(n+1)}=-\frac{1}{n}b_k^{(n)}+\frac{1}{4}b_{k-1}^{(n-1)}$$

with

$$b_k^{(n)}=0\quad\left(k<0\quad\text{or}\quad k>\left[\frac{n-1}{2}\right]\right).$$

Here $[n]$ denotes the integer part of n.

In a recent series of two papers, Petojević [9, 10] has given several new formulas for the sums of the products of Bernoulli numbers.

In [11], Petojević and Srivastava derived several formulas for the evaluation of Euler's type and Dilcher's type sums of products of Bernoulli numbers.

In this paper, we consider the special values at non-positive integers of the important representation of the multiple Hurwitz zeta-function in terms of the Hurwitz zeta-function (see (1.10) above). These special values imply closed expressions for sums of products of Bernoulli numbers of the form

$$\sum_{\substack{r_1,\dots,r_n\geq 0\\ r_1+\dots+r_n=r}} \binom{r}{r_1,\dots,r_n} B_{r_1}\cdots B_{r_n}.$$

2. Proofs of (1.4)–(1.7)

Let us put $F_n(t)=\left(\frac{t}{e^t-1}\right)^n$. Then $F_n(t)$ is the generating function of higher-order Bernoulli numbers $B_k^{(n)}$. The derivative $F_n'(t)=\frac{d}{dt}F_n(t)$ of $F_n(t)$ is equal to

$$\left(\frac{1}{t}-\frac{e^t}{e^t-1}\right)nF_n(t)=\frac{n}{t}F_n(t)-nF_n(t)-\frac{n}{e^t-1}F_n(t),$$

and $tF_n'(t)=nF_n(t)-ntF_n(t)-nF_{n+1}(t)$, so we have

$$\frac{B_k^{(n)}}{(k-1)!}=n\frac{B_k^{(n)}}{k!}-n\frac{B_{k-1}^{(n)}}{(k-1)!}-n\frac{B_k^{(n+1)}}{k!}$$

for $k \geq 1$. This formula can be written as

$$B_k^{(n+1)} = \left(1 - \frac{n}{k}\right) B_k^{(n)} - kB_{k-1}^{(n)}.$$

Thus, we can deduce (1.4).

To see (1.5), note that $B_k^{(n+1)} = \sum_{i=0}^{k} \binom{k}{i} B_i B_{k-i}^{(n)}$. Therefore from (1.4), we obtain $\sum_{i=1}^{k} \binom{k}{i} B_i B_{k-i}^{(n)} = -\frac{n}{k} B_k^{(n)} - kB_{k-1}^{(n)}$. Since all Bernoulli numbers with odd index, except B_1, equal zero, we obtain

$$-\frac{n}{k} B_k^{(n)} = \sum_{i=2}^{k} \binom{k}{i} B_i B_{k-i}^{(n)} + \frac{k}{2} B_{k-1}^{(n)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^i B_i B_{k-i}^{(n)}.$$

Thus (1.5) follows.

Note that

$$\sum_{k=0}^{\infty} B_k^{(n)}(n-x) \frac{(-t)^k}{k!} = \frac{(-t)^n e^{(x-n)t}}{(e^{-t}-1)^n} = \frac{t^n e^{xt}}{(e^t-1)^n} = \sum_{k=0}^{\infty} B_k^{(n)}(x) \frac{t^k}{k!}.$$

Equating coefficients of t^k on both sides of the above identity we obtain the desired formula (1.6).

From (1.2), we have the expression for $B_k^{(n)}(x)$,

$$B_k^{(n)}(x) = ({}^1B(x_1) + \cdots + {}^nB(x_n))^k,$$

where in the multinomial expansion of this identity we mean that

$$({}^iB(x_i))^j \text{ (the } j\text{th power of } {}^iB(x_i))\!=\!B_j(x_i),$$

which is the result (1.7) (cf. [12, Theorem 8]).

3. Main results

To derive our main theorem, we need the following lemmas.

Lemma 3.1. *Let $x \in \mathbb{R}$ and $n \geq 1$. Then*

$$\sum_{l=0}^{n-1} p_{n,l}(n) x^l = \frac{1}{(n-1)!} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (x-n)^{i-1}.$$

In particular, if $n = 1$, $p_{1,0}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$.

Proof. Note that

$$p_{n,l}(n) = \frac{1}{(n-1)!} \sum_{j=l}^{n-1} \binom{j}{l} \begin{bmatrix} n \\ j+1 \end{bmatrix} (-n)^{j-l}$$

for $l \geq 0$. Thus we find

$$(n-1)! \sum_{l=0}^{n-1} p_{n,l}(n) x^l = \sum_{j=0}^{n-1} \begin{bmatrix} n \\ j+1 \end{bmatrix} (-n)^j + \sum_{j=1}^{n-1} \binom{j}{1} \begin{bmatrix} n \\ j+1 \end{bmatrix} (-n)^{j-1} x$$
$$+ \cdots + \sum_{j=n-1}^{n-1} \binom{j}{n-1} \begin{bmatrix} n \\ j+1 \end{bmatrix} (-n)^{j-n+1} x^{n-1}.$$

Here we need to rearrange the result slightly, to $\begin{bmatrix} n \\ j+1 \end{bmatrix}, j = 1, \ldots, n$, and then note that

$$\begin{aligned}(n-1)! \sum_{l=0}^{n-1} p_{n,l}(n) x^l = \begin{bmatrix} n \\ 1 \end{bmatrix} + \begin{bmatrix} n \\ 2 \end{bmatrix} \sum_{j=0}^{1} \binom{1}{j} (-n)^{1-j} x^j \\ + \cdots + \begin{bmatrix} n \\ n-1 \end{bmatrix} \sum_{j=0}^{n-2} \binom{n-2}{j} (-n)^{n-2-j} x^j \\ + \begin{bmatrix} n \\ n \end{bmatrix} \sum_{j=0}^{n-1} \binom{n-1}{j} (-n)^{n-1-j} x^j,\end{aligned}$$

which is the desired equality. □

Lemma 3.2. *Let $n > 1$. Then*

$$\sum_{l=0}^{n-1} p_{n,l}(n) k^l = 0, \quad k = 1, \ldots, n-1.$$

Proof. Using Lemma 3.1 we note that

$$\sum_{l=0}^{n-1} p_{n,l}(n) x^l = \frac{1}{(n-1)!} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (x-n)^{i-1}. \tag{3.1}$$

Setting $x = k$ in (3.1) with $1 \leq k \leq n-1$. From (1.8), we have

$$\begin{aligned}\sum_{l=0}^{n-1} p_{n,l}(n) k^l &= \frac{1}{(k-n)(n-1)!} \sum_{i=1}^{n} \begin{bmatrix} n \\ i \end{bmatrix} (k-n)^i \\ &= 0.\end{aligned}$$

This completes the proof. □

Next we shall turn to Dilcher's type sums of products of Bernoulli numbers (cf. [3]).

Theorem 3.3.

$$\begin{aligned}&\sum_{\substack{r_1, \ldots, r_n \geq 0 \\ r_1 + \cdots + r_n = r}} \binom{r}{r_1, \ldots, r_n} B_{r_1} \cdots B_{r_n} \\ &= \begin{cases} (-1)^{n+r+1} \dfrac{r!}{(r-n)!} \displaystyle\sum_{l=0}^{n-1} p_{n,n-l-1}(n) \frac{B_{r-l}}{r-l}, & r > n \\ r! p_{r,0}(r) B_1 - r! \displaystyle\sum_{l=0}^{r-2} p_{r,r-l-1}(r) \frac{B_{r-l}}{r-l}, & r = n \\ (-1)^r r! (n-r-1)! p_{n,n-r-1}(n), & 0 \leq r \leq n-1. \end{cases}\end{aligned}$$

Proof. Case (I): $r > n$. From the definition (1.1) of the multiple Hurwitz-zeta function, we have

$$\zeta_n(s, n) = \sum_{k_1, \ldots, k_n = 1}^{\infty} (k_1 + \cdots + k_n)^{-s}.$$

In (1.3) we replace x by n with $k = r - n$ $(r > n)$ and using (1.6), this function immediately gives

$$\zeta_n(n-r,n) = (-1)^n \frac{(r-n)!}{r!} B_r^{(n)}(n) = (-1)^{n+r} \frac{(r-n)!}{r!} B_r^{(n)}. \tag{3.2}$$

On the other hand, we use (1.10) so that

$$\zeta_n(n-r,n) = \sum_{l=0}^{n-1} p_{n,l}(n) \zeta(n-r-l,n).$$

Indeed, the well-known difference equation $B_i(x+1) - B_i(x) = ix^{i-1}$ if $i \geq 2$ and $\zeta(-k,x) = -B_{k+1}(x)/(k+1)$ if $k \geq 0$ (cf. [1]) implies

$$\begin{aligned} \zeta_n(n-r,n) &= -\sum_{l=0}^{n-1} p_{n,l}(n) \frac{B_{r+l-n+1}(n)}{r+l-n+1} \\ &= -\sum_{l=0}^{n-1} p_{n,l}(n) \left(\frac{B_{r+l-n+1}(1)}{r+l-n+1} + \sum_{k=1}^{n-1} k^{r+l-n} \right), \end{aligned} \tag{3.3}$$

because $B_i(n) = B_i(1) + i \sum_{k=1}^{n-1} k^{i-1}$ for $i \geq 2$. Using Lemma 3.2, we see that

$$\sum_{l=0}^{n-1} p_{n,l}(n) \sum_{k=1}^{n-1} k^{r+l-n} = \sum_{k=1}^{n-1} \left(\sum_{l=0}^{n-1} p_{n,l}(n) k^l \right) k^{r-n} = 0. \tag{3.4}$$

Therefore, applying the known formula $\zeta(1-r,1) = -B_r(1)/r = -B_r/r$ $(r > n \geq 1)$ and (3.4) to the identity (3.3), we deduce that

$$\begin{aligned} \zeta_n(n-r,n) &= -\sum_{l=0}^{n-1} p_{n,l}(n) \frac{B_{r+l-n+1}}{r+l-n+1} \\ &= -\sum_{l=0}^{n-1} p_{n,n-l-1}(n) \frac{B_{r-l}}{r-l}. \end{aligned} \tag{3.5}$$

From (1.7), (3.2) and (3.5), Case (I) is now established.

Case (II): $r = n$. For $k \geq 0$, we know that

$$\zeta(-k) = -\frac{B_{k+1}(1)}{k+1} = \begin{cases} B_1, & k = 0 \\ -\frac{B_{k+1}}{k+1}, & k \geq 1. \end{cases} \tag{3.6}$$

Setting $r = n$ in (3.2), we obtain

$$\zeta_r(0,r) = \frac{(-1)^r}{r!} B_r^{(r)}(r) = \frac{1}{r!} B_r^{(r)}. \tag{3.7}$$

Also, by (3.4) and (3.5), we have

$$\begin{aligned} \zeta_r(0,r) &= -\sum_{l=0}^{r-1} p_{r,r-l-1}(r) \frac{B_{r-l}(1)}{r-l} \\ &= p_{r,0}(r) B_1 - \sum_{l=0}^{r-2} p_{r,r-l-1}(r) \frac{B_{r-l}}{r-l}, \end{aligned} \tag{3.8}$$

because $B_r = B_r(1)$ if $r \geq 2$. Now, combining (1.7), (3.7) and (3.8) gives Case (II).

Case (III): $0 \le r \le n-1$. The multiple Hurwitz zeta-function $\zeta_n(s,x)$ having the only singularities at $s = k$ $(k = 1, 2, \ldots, n)$ which are all simple poles with the residues

$$\operatorname*{Res}_{s=k} \zeta_n(s,n) = \frac{(-1)^{n-k}}{(k-1)!(n-k)!} B_{n-k}^{(n)}(n) \tag{3.9}$$

(cf. [2, 5]). Moreover, using (1.10), we note that

$$\begin{aligned}\operatorname*{Res}_{s=k} \zeta_n(s,n) &= \operatorname*{Res}_{s=k} \sum_{l=0}^{n-1} p_{n,l}(n)\zeta(s-l,n) \\ &= p_{n,k-1}(n)\end{aligned} \tag{3.10}$$

for $k = 1, 2, \ldots, n$, since $\zeta(s,n)$ has only simple pole at $s = 1$ with residue 1. From (3.9) and (3.10), we have

$$B_r^{(n)}(n) = (-1)^r r!(n-r-1)! p_{n,n-r-1}(n), \quad 0 \le r \le n-1. \tag{3.11}$$

From (1.7) and (3.11), we can deduce Case (III). □

Example 3.4. As a special case we state formulas for sums of products of two, respectively three, Bernoulli numbers.

(1) Consider $\zeta_2(s,2) = \sum_{k_1,k_2=1}^{\infty}(k_1+k_2)^{-s}$. In (3.2) we replace n by 2. One immediately gives

$$\begin{aligned}\zeta_2(2-r,2) &= (-1)^r \frac{1}{r(r-1)} B_r^{(2)} \\ &= (-1)^r \frac{1}{r(r-1)} \sum_{i=0}^{r} \binom{r}{i} B_i B_{r-i}.\end{aligned}$$

Also we use (3.3) with $n = 2$ so that

$$\begin{aligned}\zeta_2(2-r,2) &= \zeta(1-r,2) - \zeta(2-r,2) \\ &= -\frac{B_r(2)}{r} + \frac{B_{r-1}(2)}{r-1} \\ &= -\frac{B_r(1)+r}{r} + \frac{B_{r-1}(1)+r-1}{r-1} \\ &= -\frac{B_r}{r} + \frac{B_{r-1}}{r-1}\end{aligned}$$

for $s = r-2$ and $r > 2$. Therefore, one gets the classical identity

$$\sum_{i=2}^{r-2} \binom{r}{i} B_i B_{r-i} = ((-1)^r + 1) r B_{r-1} - ((-1)^r (r-1) + 2) B_r \quad (r \ge 4), \tag{3.12}$$

which is also known in its equivalent form

$$\sum_{k=1}^{m-1} \frac{(2m)!}{(2k)!(2m-2k)!} B_{2i} B_{2m-2k} = -(2m+1) B_{2m} \quad (m \ge 2). \tag{3.13}$$

This was found by many authors, including Euler (cf. [3, 4]). Dilchler remarked in [3] that it may be of interest to find formulas of the above type for sums of products of generalized Bernoulli numbers.

(2) Consider the multiple zeta functions $\zeta_3(s,3) = \sum_{k_1,k_2,k_3=1}^{\infty}(k_1+k_2+k_3)^{-s}$. From (3.2) we have

$$\zeta_3(3-r,3) = \frac{(-1)^{r+1}}{r(r-1)(r-2)} \sum_{\substack{i_1,i_2,i_3\geq 0\\ i_1+i_2+i_3=r}} \binom{r}{i_1,i_2,i_3} B_{i_1}B_{i_2}B_{i_3}$$

for $r > n = 3$. Also we use (3.3) with $n = 3$ so that

$$\begin{aligned}\zeta_3(3-r,3) &= \frac{1}{2}(\zeta(1-r,3) - 3\zeta(2-r,3) + 2\zeta(3-r,3)) \\ &= -\frac{1}{2}\frac{B_r(3)}{r} + \frac{3}{2}\frac{B_{r-1}(3)}{r-1} - \frac{B_{r-2}(3)}{r-2} \\ &= -\frac{1}{2}\frac{B_r}{r} + \frac{3}{2}\frac{B_{r-1}}{r-1} - \frac{B_{r-2}}{r-2},\end{aligned}$$

since $B_r(3) = B_r + r(1+2^{r-1})$. Therefore for $r > 3$ we have the sum of product of 3 Bernoulli numbers:

$$\begin{aligned}&\sum_{\substack{i_1,i_2,i_3\geq 0\\ i_1+i_2+i_3=r}} \binom{r}{i_1,i_2,i_3} B_{i_1}B_{i_2}B_{i_3} \\ &\qquad = (-1)^r r(r-1)(r-2)\left(\frac{1}{2}\frac{B_r}{r} - \frac{3}{2}\frac{B_{r-1}}{r-1} + \frac{B_{r-2}}{r-2}\right).\end{aligned}$$

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