# Higher dimensional Enriques varieties with even index

Jin Hong Kim
Department of Mathematical Sciences
Korea Advanced Institute of Science and Technology
Yuseong-gu, Daejeon 305–701, Republic of Korea
E-mail: jinkim@kaist.ac.kr

May 23, 2011

## Abstract

Let $Y$ be an Enriques variety of complex dimension $2n-2$ with $n \geq 2$ whose fundamental group is cyclic of order $n$. Assume that $n = 2m$ for prime $m$. In this paper we show that $Y$ is the quotient of a product of a Calabi-Yau manifold of dimension $2m$ and an irreducible holomorphic symplectic manifold of dimension $2m-2$ by an automorphism of order $n$ acting freely. We also show that both $Y$ and its universal cover are always projective.

Keywords: Enriques varieties, holomorphic symplectic manifolds, index

## 1 Introduction and Main results

A compact complex smooth Kähler manifold $X$ is called *irreducible symplectic* if $X$ is simply connected and $H^0(X, \Omega_X^2)$ is generated by a nowhere vanishing holomorphic 2-form. It can be considered as a higher dimensional analogue of $K3$ surfaces (see [6] for more details). Every automorphism of finite order on $K3$ surfaces without fixed points is a non-symplectic involution and their quotients are known to be the Enriques surfaces (see [1]).

Recently, in their paper [4] Boissière, Nieper-Wisskirchen, and Sarti introduced the notion of an Enriques variety which is a higher dimensional analogue of the Enriques surface. To be precise, a compact complex smooth Kähler manifold $Y$ of dimension $2n-2$ with $n \geq 2$ is called an *Enriques variety* if its canonical divisor $K_Y$ has order $n$ in the Picard group $\mathrm{Pic}(Y)$

and the holomorphic Euler characteristic $\chi(Y, \mathcal{O}_Y)$ is equal to 1. Even more generally, $Y$ is called an *intermediate Enriques variety* if for some divisor $d$ of $n$, $K_Y$ has order $d$ in $\mathrm{Pic}(Y)$ and $\chi(Y, \mathcal{O}_Y) = n/d$. As in the paper [9], the order of the fundamental group $\pi_1(Y)$ of $Y$ will be called the *index* of $Y$.

The study of automorphisms of $K3$ surfaces was essentially initiated by the work [7] of Nikulin, and since then a lot of progress has been made. One of the primary motivations to consider the Enriques variety as in our paper is, in fact, to study the automorphisms of irreducible holomorphic symplectic manifolds (see [2], [3], and [8] for some earlier works). We hope we will have another chance to deal with more on the study of automorphisms of an irreducible holomorphic symplectic manifold elsewhere in the near future.

The aim of this paper is to give some more refined results on Enriques varieties with even index which have been motivated by Theorem 3.1 in the paper [4] of Boissière, Nieper-Wisskirchen, and Sarti. To be more precise, in the second version of the paper [4] Boissière, Nieper-Wisskirchen, and Sarti proved the following theorem (Theorem 3.1 of [4] or Proposition 2.1 of [5]).

**Theorem 1.1.** *Let $Y$ be an Enriques variety of complex dimension $2n-2$ with $n \geq 2$.*

(a) *If $n$ is prime, then $Y$ is the quotient of an irreducible holomorphic symplectic manifold by a fixed point free automorphism of order $n$. In particular, $Y$ is projective and $\pi_1(Y)$ is cyclic of order $n$.*

(b) *If $n$ is odd and $\pi_1(Y)$ is cyclic of order $n$, then $Y$ is the quotient of an irreducible holomorphic symplectic manifold by a fixed point free automorphism of order $n$. In particular, $Y$ is projective.*

In the same paper [4, 5] , Boissière-Nieper-Wisskirchen-Sarti also gave a counterexample to Theorem 1.1 (b) in case when $n$ is even. Their counterexample is a 10-dimensional Enriques variety which is the quotient of a product of a 6-dimensional Calabi-Yau manifold and a 4-dimensional irreducible holomorphic symplectic manifold by an automorphism of order 6 which splits and acts freely (see Section 5.3 of [4] or Section 4.3 of [5] for more details). Roughly speaking, results of our paper show that essentially there can be no other types of counterexample in higher dimensions. More precisely, our main result is

**Theorem 1.2.** *Let $Y$ be an Enriques variety of complex dimension $2n-2$ with $n \geq 2$ whose fundamental group $\pi_1(Y)$ is cyclic of order $n$. Assume that $n = 2m$ for prime $m$. Then the following assertions hold:*

(a) *If $m$ is equal to $2$, then $Y$ is either the quotient of an irreducible symplectic holomorphic manifold of complex dimension $6$ by an automorphism $f$ of order $4$ acting freely or the quotient of a product of a Calabi-Yau manifold of complex dimension 4 and a Calabi-Yau manifold of complex dimension $2$ (or a $K3$ surface) by an automorphism $f$ of order $4$ acting freely.*

(b) *If $m$ is odd prime, then $Y$ is the quotient of a product of a Calabi-Yau manifold of complex dimension $2m$ and an irreducible holomorphic symplectic manifold of complex dimension $2m-2$ by an automorphism $f$ of order $n$ acting freely.*

*Remark* 1.3. (a) In their paper [9], Oguiso and Schröer defined an *Enriques manifold* to be a compact complex manifold that is not simply connected and whose universal cover is an irreducible holomorphic symplectic manifold. So an Enriques variety of complex dimension $2n-2$ whose fundamental group $\pi_1(Y)$ is cyclic of order $n = 2m$ with odd prime $m$ is *not* an Enriques manifold in the sense of Oguiso and Schröer.

(b) Every Enriques variety of dimension 2 is an Enriques surface in the classical sense (this is Theorem 1.2 with $m = 1$).

(c) We do not know if the automorphism $f$ in Theorem 1.2 always splits so that $Y$ decomposes into a product which already contains an Enriques variety.

In this paper, we also show the following theorem.

**Theorem 1.4.** *Let $Y$ be an Enriques variety of complex dimension $2n-2$ with $n \geq 2$ whose fundamental group $\pi_1(Y)$ is cyclic of order $n$. Assume that $n = 2m$ for prime $m$. Then both $Y$ and its universal cover are always projective.*

We organize this paper as follows. In Section 2, we shall give a proof of Theorem 1.2. Section 3 is devoted to proving Theorem 1.4.

## 2 Proof of Theorem 1.2

The goal of this section is to provide a proof of Theorem 1.2.

*Proof of Theorem 1.2.* Assume first that $m$ is odd prime. Since $K_Y$ has order $n$ in $\mathrm{Pic}(Y)$, there exists a finite unramified covering $\pi : X \to Y$ of order $n$ so that $K_X = \pi^* K_Y$ is trivial. By assumption, the fundamental group $\pi_1(Y)$ is cyclic of order $n$. So $X$ is actually the universal covering of $Y$. Since $K_X = \pi^* K_Y$ is trivial, the first Chern class $c_1(X)$ is also zero. Hence, it follows from the Bogomolov decomposition theorem of compact Kähler manifolds with $c_1 = 0$ that $X$ is isomorphic to a product

$$T \times \prod_i V_i \times \prod_j W_j, \tag{2.1}$$

where $T$ is a complex torus, $V_i$ is a Calabi-Yau manifold, and $W_j$ is an irreducible holomorphic symplectic manifold.

Now note that

$$n = \chi(X, \mathcal{O}_X) = \chi(T, \mathcal{O}_T) \prod_i \chi(V_i, \mathcal{O}_{V_i}) \times \prod_j \chi(W_j, \mathcal{O}_{W_j}),$$

where we used $\chi(X, \mathcal{O}_X) = n \times \chi(Y, \mathcal{O}_Y) = n$ in the first equality. But, if $\dim_{\mathbb{C}} V_i$ is odd, then $\chi(V_i, \mathcal{O}_{V_i}) = 0$. Hence $\dim_{\mathbb{C}} V_i$ is even. Moreover, if $\dim_{\mathbb{C}} T$ is greater than 1, then $\chi(T, \mathcal{O}_Y) = 0$. On the other hand, if $\dim_{\mathbb{C}} T$ is equal to 1, then the complex dimension of $X$ not only becomes odd but also is not simply connected. Therefore, there are no complex torus and no Calabi-Yau manifolds with odd complex dimension in the decomposition (2.1). Recall also the following well-known fact that

$$\chi(W_j, \mathcal{O}_{W_j}) = \frac{\dim_{\mathbb{C}} W_j}{2} + 1 \geq 2.$$

With theses understood, we now obtain

$$n = 2m = \chi(X, \mathcal{O}_X) = 2^t \left(\frac{w_1}{2} + 1\right) \cdots \left(\frac{w_s}{2} + 1\right), \tag{2.2}$$

where $t$ denotes the number of Calabi-Yau manifolds of even complex dimension in the decomposition (2.1) and $w_j$ denotes the complex dimension of $W_j$. Since $m$ is assumed to be an odd prime, it follows from (2.2) that $0 \leq t \leq 1$. If $t$ is equal to 1, then we have

$$\begin{gathered} m = \left(\frac{w_1}{2} + 1\right) \cdots \left(\frac{w_s}{2} + 1\right) \\ 2n - 2 = 4m - 2 = v_1 + w_1 + \cdots + w_s, \end{gathered} \tag{2.3}$$

where $v_i$ denotes the complex dimension of $V_i$. Since $m$ is an odd prime, $s$ is equal to 1 by the first equation of (2.3), and $v_1 = 2m$ and $w_1 = 2m - 2$. On the other hand, if $t$ is equal to 0, then there exists an $i$, say 1, such that

$\frac{w_i}{2} + 1 = 2$, and $s = 2$. Moreover, it is easy to obtain that $4m - 4 = w_2 = 2m - 2$, which implies $m = 1$. This is a contradiction. This completes the proof of Theorem 1.2 (b).

Next, we deal with the case when $m = 2$. To do so, we first consider the case when $t$ is equal to 2. Then it follows from (2.2) that we have $s = 0$. By taking into account the dimension of $X$, we also have $v_1 + v_2 = 6$. Since $t$ is the number of Calabi-Yau manifolds of even complex dimension, there are only two possibilities for $v_1$ and $v_2$: either $v_1 = 2$ and $v_2 = 4$ or $v_1 = 4$ and $v_2 = 2$. In either case, $Y$ is the quotient of a product of a Calabi-Yau manifold of complex dimension 2 (or a $K3$ surface) and a Calabi-Yau manifold of complex dimension 4 by an automorphism $f$ of order 4 acting freely.

On the other hand, if $t$ is equal to 1, we have $w_1 = 2$ by (2.2), and $v_1 = 4$ by the second equation of (2.3). So $Y$ is the quotient of a product of a Calabi-Yau manifold of complex dimension 4 and a Calabi-Yau manifold of complex dimension 2 (or a $K3$ surface) by an automorphism $f$ of order 4 acting freely. Recall that a Calabi-Yau manifold of complex dimension 2 is the same as an irreducible holomorphic symplectic manifold of the same dimension, and, in fact, they are all $K3$ surfaces.

Finally, if $t$ is equal to 0, then it follows from (2.2) that we have two possibilities for $s$: either $s = 1$ or $s = 2$. If $s$ is equal to 1, then we have $w_1 = 6$ by (2.2). In this case, $Y$ is simply the quotient of an irreducible symplectic holomorphic manifold of complex dimension 6 by an automorphism $f$ of order 4 acting freely. If $s$ is equal to 2, then we have $w_1 = w_2 = 2$ by (2.2) and thus $w_1 + w_2 = 4$. But this is not equal to the complex dimension of $X$ that is equal to 6. So this case does not occur. This completes the proof of Theorem 1.2 (a).

This completes the proof of Theorem 1.2. $\square$

## 3 Proof of Theorem 1.4

In this section, we give a a proof of Theorem 1.4.

*Proof of Theorem 1.4.* Let $h^{p,q}(Y) = \dim_{\mathbb{C}} H^q(Y, \Omega_Y^p)$ and let $G$ be the cyclic group generated by the automorphism $f$ in Theorem 1.2. Then it is clear that $H^0(Y, \Omega_Y^p) = H^0(X, \Omega_X^p)^G$. The proof is divided into three steps:

*Step 1:* In this step, we first deal with the case that $Y$ is the quotient of a product of a Calabi-Yau manifold $V$ of complex dimension $2m$ and an irreducible holomorphic symplectic manifold $W$ of complex dimension $2m-2$ by an automorphism $f$ of order $n$ acting freely. We then show that $H^0(Y, \Omega_Y^2) = 0$. This will be a key ingredient in *Step 3* to prove that $Y$ and its universal cover $X$ are projective.

To prove that $H^0(Y, \Omega_Y^2) = 0$, note first from the Künneth formula that we have

$$H^0(X, \Omega_X^p) = \bigoplus_{r+s=p} H^0(V, \Omega_V^r) \otimes H^0(W, \Omega_W^s).$$

Recall then that for a Calabi-Yau manifold $V$, $h^{r,0}(V) = 0$ for $0 < r < \dim_{\mathbb{C}} V$, while for an irreducible holomorphic manifold $W$, $h^{s,0}(W) = 0$ for odd $s$ with $0 < s < \dim_{\mathbb{C}} W$. Thus, it is easy to obtain that for $\dim_{\mathbb{C}} V \leq p \leq \dim_{\mathbb{C}} V + \dim_{\mathbb{C}} W = \dim_{\mathbb{C}} X$,

$$\begin{aligned} H^0(X, \Omega_X^p) &= H^0(V, \Omega_V^{\dim_{\mathbb{C}} V}) \otimes H^0(W, \Omega_W^{p-\dim_{\mathbb{C}} V}) \\ &= \begin{cases} \mathbb{C}, & \text{for even } p \text{ with } \dim_{\mathbb{C}} V \leq p \leq \dim_{\mathbb{C}} V + \dim_{\mathbb{C}} W, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.1}$$

On the other hand, for $0 \leq p \leq \dim_{\mathbb{C}} W$ we have

$$\begin{aligned} H^0(X, \Omega_X^p) &= H^0(W, \Omega_W^p) \\ &= \begin{cases} \mathbb{C}, & \text{for even } p \text{ with } 0 \leq p \leq \dim_{\mathbb{C}} W, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2}$$

The fact that $\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W + 2$ will also play an important role later.

Next we claim that $h^{2,0}(Y) = h^{0,2}(Y) = 0$. Indeed, it follows from the equation (3.2) and $\dim_{\mathbb{C}} W \geq 2$ that $H^0(X, \Omega_X^2) = H^0(W, \Omega_W^2)$. So let $\sigma$ be a generator of $H^0(W, \Omega_W^2)$. Then the following lemma holds.

**Lemma 3.1.** *$f$ is actually non-symplectic.*

*Proof.* To prove it, suppose that, on the contrary, $f$ is not non-symplectic. Then there would exist some integer $i$ ($1 \leq i \leq n-1$) such that $f^i$ is symplectic, i.e., $(f^i)^*\sigma = \sigma$. Without loss of generality, we assume that $i$ is equal to 1. Since $H^0(W, \Omega_W^p)$ is zero for odd $p$ and generated by $\sigma^{p/2}$ for even $p$, it is easy to obtain

$$\sum_{p=0}^{\dim_{\mathbb{C}} W} (-1)^p \mathrm{tr}\left(f^*|_{H^0(W, \Omega_W^p)}\right) = \frac{\dim_{\mathbb{C}} W}{2} + 1. \tag{3.3}$$

Now, let $\Theta_V$ denote a generator of $H^0(V, \Omega_V^{\dim_{\mathbb{C}} V})$. Then we see from (3.1) that there exists some $x \in \mathbb{C}$ such that

$$f^*(\Theta_V) = x\Theta_V.$$

Since we have

$$f^*(\Theta_V \otimes \sigma^k) = x\Theta_V \otimes \sigma^k$$

for all $0 \leq k \leq \frac{\dim_{\mathbb{C}} W}{2}$, we obtain

$$\begin{aligned} &\sum_{p=\dim_{\mathbb{C}} V}^{\dim_{\mathbb{C}} V+\dim_{\mathbb{C}} W} (-1)^p \mathrm{tr}\left(f^*|_{H^0(V,\Omega_V^{\dim_{\mathbb{C}} V})\otimes H^0(W,\Omega_W^{p-\dim_{\mathbb{C}} V})}\right) \\ &= x\left(\frac{\dim_{\mathbb{C}} W}{2}+1\right). \end{aligned} \tag{3.4}$$

Now, if we combine two equations (3.3) and (3.4), it is immediate to obtain the holomorphic Lefschetz number $L(f)$ of $f$ that is equal to

$$L(f) = \sum_{p=0}^{\dim_{\mathbb{C}} X} (-1)^p \mathrm{tr}\left(f^*|_{H^0(X,\Omega_X^p)}\right) = (1+x)\left(\frac{\dim_{\mathbb{C}} W}{2}+1\right).$$

At this point, it is important to notice that the only case of symplectic $f$ which would give a vanishing holomorphic Lefschetz number is when $x$ is equal to $-1$. More precisely, if $x$ is not equal to $-1$, then the holomorphic Lefschetz number $L(f)$ of $f$ is not zero. So $f$ would have a fixed point. But, this contradicts the fact that $f$ acts freely.

Therefore, we assume that $x$ is equal to $-1$. Then, by using the same argument as above it is easy to see that the holomorphic Lefschetz number $L(f^2)$ of $f^2$ would be equal to

$$2\left(\frac{\dim_{\mathbb{C}} W}{2}+1\right) = \dim_{\mathbb{C}} W + 2,$$

which is clearly non-zero. So $f^2$ would have a fixed point. However, again this contradicts the fact that $f$ acts freely. This completes the proof of Lemma 3.1. $\square$

Consequently, by Lemma 3.1 we have $f^*\sigma = \xi\sigma$ for some $n$-th root of unity $\xi$. But then clearly $\sigma$ is not invariant under $f$. Hence we have $H^0(Y, \Omega_Y^2) = 0$. This completes the proof of *Step 1.*

*Step 2:* In this step, we next consider the case that $Y$ is the quotient of an irreducible symplectic holomorphic manifold $X$ of complex dimension 6 by an automorphism $f$ of order 4 acting freely. In the same way as in *Step 1*, we can show that $H^0(Y, \Omega_Y^2) = 0$, which enables us to show that $Y$ and its universal cover $X$ are projective in *Step 3*.

Indeed, note first that we have

$$H^0(X, \Omega_X^p) = \begin{cases} \mathbb{C}, & \text{for even } p \text{ with } 0 \leq p \leq \dim_{\mathbb{C}} X = 6, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\sigma$ be a generator of $H^0(X, \Omega_X^2)$. Then it can be shown exactly as in *Step 1* that there exists a primitive 4-root of the unity $\xi$ such that $f^*(\sigma) = \xi\sigma$, i.e., $f$ is non-symplectic. This implies that $\sigma$ is not invariant under $f$ (refer to Section 3.1 of [4] or Secion 2.2 of [5]). Therefore we have $H^0(Y, \Omega_Y^2) = 0$. This completes the proof of *Step 2*.

*Step 3:* Finally we prove that both $Y$ and its universal cover $X$ are projective. To do so, since $h^{2,0}(Y) = h^{0,2}(Y) = 0$ by *Step 1* and *Step 2*, note first that the inclusion of $H^{1,1}(Y)_{\mathbb{R}}$ into $H^2(Y, \mathbb{R})$ is bijective. The fact that the Kähler cone inside $H^{1,1}(Y)_{\mathbb{R}}$ is non-empty and open then implies that there exists an integral class on $Y$. Hence it follows from the Kodaira embedding theorem that the Kähler manifold $Y$ is projective. By pulling back the integral class on $Y$, we can also obtain an integral Kähler class on $X$, proving that $X$ is also projective.

This completes the proof of Theorem 1.4. □

**Acknowledgements:** This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0001651, 2011-0001181). We also want to acknowledge the support by the second stage of the Brain Korea 21 Project, The Development Project of Human Resources in Mathematics, KAIST in 2011.

# References

[1] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Ergeb. Math. Grenzgebiete **4**, Springer, Berlin, 1984.

[2] S. Boissière, *Automorphismes naturels de l'espace de Douady de points sur une surface*, to appear in Canad. J. Math.; arXiv:0905.4367v1.

[3] S. Boissière and A. Sarti, *Automorphismes de l'espace de Douady de points sur une surface* $K3$, preprint (20009); arXiv:0905.4370v1.

[4] S. Boissière, M. Nieper-Wisskirchen, and A. Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties*, preprint (2010); arXiv:1001.4728v2.

[5] S. Boissière, M. Nieper-Wisskirchen, and A. Sarti, *Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties*, to appear in Journal de Mathématiques Pures et Appliquées; arXiv:1001.4728v3.

[6] M. Gross, D. Huybrechts, and D. Joyce, *Calabi-Yau manifolds and related geometries*, Springer, Berlin, 2003.

[7] V. Nikulin, *Finite automorphism groups of Kähler* $K3$ *surfaces*, Trans. Mosc. Math. Soc. **2** (1980), 71–135.

[8] K. Oguiso, *Tits alternative in hyperkähler manifolds*, Math. Res. Lett. **13** (2006), 307–316.

[9] K. Oguiso and S. Schröer, *Enriques manifolds*, to appear in J. Reine Angew. Math.; arXiv:1001.4912v1.