ON THE TATE MODULE OF A NUMBER FIELD

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Abstract. We investigate the order of vanishing of p-adic L function at $s = 0$ via class field theory. Our investigation gives rise to relations with Tate modules of number fields and connections to conjectures of Gross and Coates-Lichtenbaum as well as a question of Kato. We reduce the conjecture to the study of universal norm elements and norm comparable elements over the cyclotomic Z_p-extension. Based on examples and technical evidences, we are lead to a conjecture on the Galois invariants of Tate module which is stronger than the Gross conjecture.

1. Introduction

Let k be a CM-field with maximal totally real subfield k^+. For an odd prime p, let $r(k)$ be the number of primes dividing p in k^+ which split in k/k^+. Gross [?] found a criterion for the order of vanishing at $T = 0$ of a generator $f_k^-(T)$ of the characteristic ideal of the inverse limit of the groups of minus ideal classes of k_n in the cyclotomic $\mathbb{Z}_p$-extension $k_\infty^{\mathrm{cyc}} = \bigcup_n k_n$ of k to be equal to $r(k)$. Iwasawa [?] and Greenberg [?] have shown that the order is greater than or equal to $r(k)$ and when k is abelian Greenberg established equality. Gross conjectured(cf. [?] and [?]), the order of vanishing of $f_k^-(T)$ is exactly equal to $r(k)$ for arbitrary CM-fields k, or equivalently, the p-adic rank of completion of the group generated by the image of the p-units under the composite of norm map and the p-adic logarithm is equal to one less than the number of primes dividing p in k^+. Using the generalization of Iwasawa's main conjecture proven by Wiles [?], this is equivalent to the statement that the order of vanishing of the p-adic L-function $L_p(\chi\omega, s)$ (see §2) at $s = 0$ is equal to $r(k)$.

The Gross conjecture investigated in here is a generalization formulated by Jaulent [?] from CM-fields to arbitrary number fields. If we denote by δ_k the deviation of the p-adic rank above from one less than the number of primes dividing p in k^+, then the Gross conjecture(cf. [?] and [?])for an arbitrary field k is

Gross conjecture: $\delta_k = 0$.

The generalization above is also explained by Kato in his paper [?]. We briefly summarize it. Let $T_p(k)$ be the Tate module for k which is defined as the inverse limit of the p-primary parts of the class groups modulo the subgroup generated by the classes of the primes dividing p over $k_\infty^{\mathrm{cyc}}/k$. Note that this notion is slightly different from the usual Tate module which is defined as the inverse limit of the p-primary parts of the ideal class groups of k_n. For a place v of k lying over p, let $G(k_\infty^{\mathrm{cyc}}/k)_v \subset G(k_\infty^{\mathrm{cyc}}/k)$ be the decomposition group of v. Let $(\oplus_{v|p} G(k_\infty^{\mathrm{cyc}}/k)_v)^0$

be the kernel of the canonical map $\oplus_{v|p} G(k_\infty^{\mathrm{cyc}}/k)_v \to G(k_\infty^{\mathrm{cyc}}/k)$. Let

$$\alpha_k : U_k(p) \otimes \mathbb{Z}_p/(U_k(p)^{\mathrm{loc}} \otimes \mathbb{Z}_p)^{\mathrm{coh}} \longrightarrow (\oplus_{v|p} G(k^{\mathrm{cyc}}/k)_v)^0$$

be the homomorphism induced by the reciprocity maps of local fields k_v. Notice the denominator was changed from Kato's $(U_k(p) \otimes \mathbb{Z}_p)^{\mathrm{coh}}$ to $(U_k(p)^{\mathrm{loc}} \otimes \mathbb{Z}_p)^{\mathrm{coh}}$ which was explained in §3. The question made by Kato is whether the following equivalent statements hold for an arbitrary number field k:
(1) $\mathrm{Ker}(\alpha_k)$ is finite. That is, $(U_k(p)^{\mathrm{loc}} \otimes \mathbb{Z}_p)^{\mathrm{coh}}$ is of finite index in the kernel of $U_k(p) \otimes \mathbb{Z}_p \to (\oplus_{v|p} G(k_\infty^{\mathrm{cyc}}/k)_v)^0$.
(2) $\mathrm{Coker}(\alpha_k)$ is finite.
(3) $H_0(G(k_\infty^{\mathrm{cyc}}/k), T_k(p))$ is finite.
Note that the equivalent statements above contains Conjecture 2.2 of Coates and Lichtenbaum [?] which says that the minus part of the 0-dimensional homology group $H_0(G(k_\infty^{\mathrm{cyc}}/k), T_k(p))^-$ is finite for the case of CM-field.

The main result of this paper is to find an equivalent condition for the Gross conjecture above. We also find a criterion for the Galois invariants $T_p(k)^\Gamma$ of $T_p(k)$ to be trivial in terms of universal norm and norm comparable conditions which will lead to a strong condition for the Gross conjecture, i.e., $H_0(G(k_\infty^{\mathrm{cyc}}/k), T_k(p))$ is trivial. For a prime number p, let $\Lambda = \varprojlim_n \mathbb{Z}_p[G(k_n/k)]$ denote the Iwasawa algebra for k_∞^{cyc} of a number field k. Then it is well known that the Tate module $T_p(k)$ is a finitely generated compact torsion-Λ-module. We will describe via class field theory the Galois invariants $T_p(k)^\Gamma$ of $T_p(k)$ which consists of elements of $T_p(k)$ fixed by $\Gamma = G(k_\infty^{\mathrm{cyc}}/k)$, and Galois coinvariants $T_p(k)_\Gamma$ of $T_p(k)$ which is the maximal quotient of $T_p(k)$ such that Γ acts trivially. Then, the Gross conjecture described above can be reincarnated via the class field theory in the condition of finiteness of the Galois coinvariants of the Tate module for $k_\infty^{\mathrm{cyc}}/k$.

In §2, we will introduce the Gross conjecture and briefly explain the role of conjecture in both defining a p-adic L-function and studying its arithmetic properties at origin. In §3, we will reformulate the Gross conjecture via the infinite class field theory(cf. [?] and [?]) for which we refer papers by Kuz'min [?], Sinnott [?] and others. Kuz'min describes the invariants of the Tate module in terms of universal local and global elements, while Sinnott describes the coinvariants of the Tate module using the p-adic logarithm map. Our reformulation of the Gross conjecture then leads to a problem on certain global norm comparable properties over the cyclotomic $\mathbb{Z}_p$-extension by making use of these results. For a finite set A, we denote by $\#A$ the cardinality of A. For a number field F with its cyclotomic $\mathbb{Z}_p$-extension $F_\infty^{\mathrm{cyc}} = \bigcup_n F_n$, let $\varprojlim_n F_n^\times$ denote the inverse limit of $F_n^\times$ with respect to the norm maps. For the intermediate field F_n, let $U_n(p)$ be the group of p-units of F_n. Let

$$F^{\mathrm{univ}} = \bigcap_n N_{F_n/F} F_n^\times \text{ and } F^{\mathrm{coh}} = \pi(\varprojlim_n F_n)$$

where π denotes the natural projection from $\varprojlim_n F_n^\times$ to $F^\times$ and $N_{L/K}$ denotes the norm map for an extension L/K. By replacing the field F with the group $U_F(p)$ of p-units of F, we have similar notions $U_F(p)^{\mathrm{univ}} = \bigcap_n N_{F_n/F} U_n(p)$ and $U_F(p)^{\mathrm{coh}} = \pi(\varprojlim_n U_n(p))$. By replacing the field F with intermediate field F_n, we have that $U_{F_n}(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{univ}} = \bigcap_{m \geq n} N_{F_m/F_n} U_m(p)$ and $U_{F_n}(p)^{\mathrm{coh}} = U_n(p)^{\mathrm{coh}} = \pi(\varprojlim_{m \geq n} U_m(p))$. By considering ramifications over the cyclotomic

$\mathbb{Z}_p$-tower, we notice that $F^{\text{univ}} = U_F(p)^{\text{univ}}$ and $F^{\text{coh}} = U_F(p)^{\text{coh}}$. For a number field k, let $T_p(k)$ be the Tate module for k_∞^{cyc}/k.

Theorem 1.1. *Let k be a number field with $k_\infty^{\text{cyc}} = \bigcup_n k_n$. Then $T_p(k)^\Gamma = 1$ if and only if $N_{k_n/k} U_n(p)^{\text{univ}} = U_k(p)^{\text{univ}}$ for all $n > 0$.*

Based on Theorem 1.1, we make the following conjecture on the Tate module of an arbitrary number field k.

Conjecture. *For a number field k, the Galois invariants of Tate module is trivial.*

$$T_p(k)^\Gamma = 1.$$

Note that this conjecture is stronger than the Gross conjecture. However, the evidence can be found in the following technical point which seems very plausible.

$$T_p(k)^\Gamma = 1 \Longleftrightarrow N_{k_n/k} U_n(p)^{\text{univ}} = U_k(p)^{\text{univ}} \text{ for all } n \geq 0.$$

We do not find any direct method which leads to $T_p(k)^\Gamma = 1$. Instead, we will find easily many examples satisfying RHS above when either k is abelian using Sinnott's circular units or p does not split in k. Hence, these examples will provide us cases for which $T_p(k)^\Gamma = 1$. For any field k, we also have the following implication.

$$U_k(p)^{\text{univ}} = U_k(p)^{\text{coh}} \Longrightarrow T_p(k)^\Gamma = 1.$$

Proposition 1.2. *Suppose the Gross conjecture is true for the intermediate fields k_n of k_∞^{cyc} over a number field k. Then*

$$U_k(p)^{\text{univ}} = U_k(p)^{\text{coh}} \Longleftrightarrow T_p(k)^\Gamma = 1.$$

Over the $\mathbb{Z}_p$-tower $k_\infty^{\text{cyc}} = \bigcup_n k_n$, we have the following equivalent condition.

Corollary 1.3. *Let k be a number field with $k_\infty^{\text{cyc}} = \bigcup_n k_n$. Then*

$$U_n(p)^{\text{univ}} = U_n(p)^{\text{coh}} \textit{ for all } n \geq 0 \Longleftrightarrow T_p(k_n)^{\Gamma_n} = 1 \textit{ for all } \ n \geq 0.$$

For the Gross conjecture, we have the following equivalent form.

Proposition 1.4. *Let k be a number field with $k_\infty^{\text{cyc}} = \bigcup_n k_n$. Then the Gross conjecture is true for k if and only if there is $n = n(k) \geq 0$ such that*

$$N_{k_{n+i}/k} U_{n+i}(p)^{\text{univ}} = N_{k_n/k} U_n(p)^{\text{univ}} \textit{ for all } i \geq 0.$$

Proposition 1.5. *The following conditions are equivalent.*

$$(U_n^{\text{univ}} : U_n^{\text{coh}}) < \infty, \textit{ for all } n \geq 0 \iff \textit{The Gross conjecture is true for all } k_n$$

The Gross conjecture was reduced to the size of the Galois invariants of the Tate module. Naturally, we can compare the structure of the Galois invariants of the Tate module with that of the quotient of the group of global universal norm elements by the group of global norm comparable elements. This observation leads us to the following guess. For a number field k, there exists an isomorphism of abelian groups.

$$T_p(k)^\Gamma \cong \frac{U_k(p)^{\text{univ}}}{U_k(p)^{\text{coh}}}.$$

The following proposition shows that the Gross conjecture implies the isomorphism above.

Proposition 1.6. *Let k be a number field such that $\{k_n\}_n$ satisfy the Gross conjecture. Then we have an isomorphism of abelian groups.*

$$T_p(k)^\Gamma \cong \frac{U_k(p)^{\mathrm{univ}}}{U_k(p)^{\mathrm{coh}}}.$$

More precisely, we will show that the Gross conjecture for all intermediate fields k_n with $n \in \{0\} \cup \mathbb{N}$ of k_∞^{cyc} implies the conjecture of structure for all k_n of k_∞^{cyc}.

2. Gross conjecture

In this section, we introduce the Gross conjecture and its reformulation in terms of the Tate module. For an arbitrary number field k, let S_k denote the set of all infinite primes and all the primes of k diving p. We denote by $U_k(p)$ the group of S_k-units of k and by $\mathrm{Cl}_k^s \otimes \mathbb{Z}_p$ the p-primary part of the S-class group Cl_k^s of k. For each finite prime v of k, let k_v^{univ} denote the group of universal norms from the cyclotomic $\mathbb{Z}_p$-extension $k_{v\infty}^{\mathrm{cyc}}$ of the local field k_v. Let $\log_p$ denote Iwasawa's p-adic logarithm normalized by $\log_p(p) = 0$. Thus, it induces an isomorphism from $\mathbb{Q}_p^\times/p^{\mathbb{Z}}\mu(\mathbb{Q}_p) = 1 + p\mathbb{Z}_p$ to $p\mathbb{Z}_p$ where $\mu(\mathbb{Q}_p)$ denotes the torsion subgroup of $\mathbb{Q}_p$. It also follows that for each $v|p$, $a \in k_v^{\mathrm{univ}}$ if and only if $a \in \mathrm{Ker}(\log_p \circ N_{k_v/\mathbb{Q}_p})$. Let $\tilde{g}_k = \sum_{v|p} \log_p \circ N_{k_v/\mathbb{Q}_p} : \prod_{v|p} k_v^\times \to \oplus_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v \cong \prod_{v|p} k_v^\times/k_v^{\mathrm{univ}}$. We will restrict the domain into k by the diagonal imbedding. The closure of the image $\tilde{g}_k(k)$ is thus equal to $\oplus_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v$. We let g_k denote the map $\tilde{g}_k$ restricted in $U_k(p)$. If no confusion occurs, then we will use the notation g_k for various maps induced from g_k by the scalar extensions. For example, we denote by g_k the homomorphism of $\mathbb{Z}_p$-modules

$$g_k : U_k(p) \otimes \mathbb{Z}_p \longrightarrow \oplus_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v$$

induced by the scalar extension by $\mathbb{Z}_p$, and the homomorphism of $\mathbb{Q}_p$-modules, $g_k : U_k(p) \otimes \mathbb{Q}_p \longrightarrow \oplus_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v$ induced by the scalar extension by $\mathbb{Q}_p$ and etc. Let d_k denote the number of primes of k dividing p. Since the $\mathbb{Z}_p$-rank $\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Im}(g_k))$ of the image of g_k is less than or equal to $d_k - 1$, we can define a certain nonnegative integer $\delta_k \geq 0$ such that

$$\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Im}(g_k)) = d_k - 1 - \delta_k.$$

Notice that by Dirichlet's unit theorem, $\mathbb{Z}_p$-rank $\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Ker}(g_k))$ of the kernel of g_k is given by

$$\mathrm{rk}_{\mathbb{Z}_p}(\mathrm{Ker}(g_k)) = r_1 + r_2 + \delta_k$$

where r_1 and r_2 denote respectively the number of real and complex primes of k. When k is a CM-field with maximal real subfield k^+, the original conjecture of Gross is that the following induced map

$$g_k^- : U_k(p)^- \otimes \mathbb{Z}_p \longrightarrow \oplus_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v$$

has finite kernel where $U_k(p)^-$ denotes the kernel of the norm map N_{k/k^+} from $U_k(p)$ to $U_k(p)^+ = U_k(p) \bigcap k^+$(cf. [**?**] and [**?**]). Let $M = M_k$ be free abelian group generated by the divisors of S_k and $M^- = \{m \in M | \tau(m) = -m\}$ be the -1-eigenspace of of M by the complex conjugation $\langle \tau \rangle = G(k/k^+)$. Then, $\mathbb{Q}_p \otimes_{\mathbb{Z}} M^-$ has $\mathbb{Q}_p$-basis $\{v - \tau(v)\}$. The conjecture can be stated as the induced map

$$g_k : U_k(p)^- \otimes \mathbb{Q}_p \longrightarrow M_k^- \otimes \mathbb{Q}_p$$

is always an isomorphism. Let ψ denote the map $\psi : \mathbb{Q}_p \otimes_{\mathbb{Z}} M^- \to \mathbb{Q}_p \otimes_{\mathbb{Z}} U_k(p)^-$ defined as

$$\psi(v - \tau(v)) = \frac{1}{h_k f_v}(\alpha - \tau(\alpha))$$

where h_k is the class number of k, f_v is the residue class degree at v and $\mathfrak{b}^{h_k} = (\alpha)$ for the corresponding prime ideal $\mathfrak{b}$ to the divisor v. It can be shown that the induced map g_k and ψ are inverse each other over $\mathbb{Q}\otimes_{\mathbb{Z}} M^-$ and $\mathbb{Q}\otimes_{\mathbb{Z}} U_k(p)^-$(cf. [?]). Define the regulator of k

$$R_k = \det(g_k\psi|_{\mathbb{Q}_p\otimes_{\mathbb{Z}} M^-})$$

to be the determinant of the endomorphism $g_k \circ \psi$ over the $\mathbb{Q}_p$-module, $\mathbb{Q}_p \otimes_{\mathbb{Z}} M^-$. Then the Gross conjecture is equivalent to

$$R_k \neq 0.$$

This condition is often related with vanishing order of p-adic L-functions(cf. [?] and [?]). We denote by $A_\infty = \lim_n(\mathrm{Cl}_n \otimes \mathbb{Z}_p)$ the direct limit of the p-primary parts of the ideal class groups of k_n. Let $f_k^-(T)$ denote a generator of the characteristic ideal of the Pontryagin dual $\mathrm{Hom}_{\mathbb{Z}_p}(A_\infty^-, \mathbb{Q}_p/\mathbb{Z}_p)$ of A_∞^- which is isomorphic to $\mathrm{Cl}_\infty(k)^- = \varprojlim(\mathrm{Cl}_n \otimes \mathbb{Z}_p)^-$. For each prime $\mathfrak{p}$ of k dividing p, let $\mathfrak{a}_{n,\mathfrak{p}} = \prod_{\mathfrak{B}|\mathfrak{p}} \mathfrak{B}$ denote the product of all primes dividing $\mathfrak{p}$ in k_n and let $\mathfrak{b}_{n,\mathfrak{p}} = \mathfrak{a}_{n,\mathfrak{p}}^{1-\tau}$. Let D_n^- denote the subgroup $\langle \mathrm{cl}(\mathfrak{b}_{n,\mathfrak{p}})\rangle$ of A_n^- generated by the classes $\mathrm{cl}(\mathfrak{b}_{n,p})$ as $\mathfrak{p}$ varies primes of k^+ dividing p which splits in k/k^+. Let $D_\infty^- = \lim D_n^-$. If we denote a generator of the characteristic ideal of $A'^-_\infty = A_\infty^-/D_\infty^-$ by $G_k(T)$, then $f_k^-(T) = T^{r(k^+)}G_k(T)$ where $r(k^+)$ denotes the number of primes of k^+ dividing p which splits in k/k^+. In this setting, the Gross conjecture is equivalent to the non vanishing of $G_k(T)$ at 0(cf. [?]).

$$G_k(0) \neq 0.$$

The extension the Gross conjecture for CM-fields to arbitrary number fields is due to Jaulent who also showed that the conjecture is true for abelian number fields(cf. [?]). Following Jaulent (cf. [?]), the Gross-conjecture is stated as follows.

Gross conjecture: $\delta_k = 0$.

Notice that for a CM-field k, if we put $\delta_k^- = \delta_k - \delta_{k^+}$, then $\delta_k^- = 0$ is equivalent with Gross original conjecture and equivalent with $\mathrm{Ker}(g_k)^- = \mu(k)$. Finally, we introduce the following equivalence of Gross conjecture. For the proof of the following known proposition, see Proposition of 1.2 of [?] or Proposition 6.5 of the appendix of [?] by Sinnott. We briefly sketch the proof for the convenience of the reader.

Proposition 2.1.

$$\delta_k = 0 \quad \textit{if and only if} \quad \#T_p(k)_\Gamma < \infty.$$

Sketch of Proof(cf. [?] and [?]). The map g_k produce the following exact sequence.

$$0 \to \mathrm{Ker}(g_k) \to U_k(p) \otimes \mathbb{Z}_p \xrightarrow{g_k} \overline{g_k(k)} \to G(L'_0/k) \to G(H'/k) \to 1$$

where H' is the p-Hilbert class field corresponding to the p-Sylow subgroup of the p-class group Cl_k^s of k and L'_0 is the maximal abelian extension of k such that L'_0/k_∞ is unramified outside p and splits completely at primes dividing p and $\overline{g_k(k)}$ denotes the closure of $g_k(k)$ in the idèle-topology. It follows from Dirichlet's unit theorem and $\overline{g_k(k)} = \sum_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times)v = \sum_{v|p} k_v^\times/\hat{U}_v$ that $\mathrm{rk}_{\mathbb{Z}_p} G(L_0'/k) = \delta_k + 1$. Hence, it follows that $\delta_k = 0$ if and only if $T_p(k)_\Gamma = G(L_0'/k_\infty) < \infty$. □

Notice that since $T_p(k)$ is a finitely generated compact torsion-Λ-module, the Gross conjecture is equivalent to

$$\# T_p(k)^\Gamma < \infty.$$

By taking Pontryagin dual, the Gross conjecture is also equivalent to

$$\# A_\infty^\Gamma < \infty.$$

The condition above is also equivalent that the following order

$$\# \mathrm{Cl}_n^{G(k_n/k)}$$

is bounded independent of n(cf. Theorem 1.14 of [?]). For a CM-field k and $G(k/k^+)$-module M, we let M^- denote the -1-eigenspace of M by the action of the complex conjugation. Let $T_p(k)$ be the Tate module for $k_\infty^{\mathrm{cyc}}/k$. Let $f_k^-(T)$ denote a generator of the characteristic ideal of $T = \gamma_u$ on $\varprojlim_n \mathrm{Hom}_{\mathbb{Z}_P}((\mathrm{Cl}_n \otimes \mathbb{Z}_p)^-, \mathbb{Q}_p/\mathbb{Z}_p)$. If k is a CM-field which satisfies the Gross conjecture, then

$$\lim_{s\to 0} \frac{f_k^-(T)}{T^{r(k)}} \neq 0.$$

For an algebraic closure $\overline{k}^+$ of k^+, let $\chi : G(\overline{k}^+/k^+) \to \pm 1$ be the odd Galois character corresponding to $G(k/k^+)$ and ω be the Teichmüller character. Let $L_p(\chi\omega, s)$ be its p-adic L-function(cf. [?] and [?]). $L_p(\chi\omega, s)$ is related to the Artin L-function $L(\chi, s)$ as follows.

$$L_p(\chi\omega, n) = \prod_{\mathfrak{p}|p} (1 - \chi(\mathfrak{p})\mathbb{N}\mathfrak{p}^{-n}) L(\chi, n) \quad n \leq 0, \quad n \equiv 0 (\mathrm{mod}\, p-1).$$

For each topological generator γ_u, with $u \in 1 + p^e\mathbb{Z}_p = G(k_\infty/k)$, there is a unique integral power series $F(T) \in \mathbb{Z}_p[[T]]$ such that

$$L_p(\chi\omega, s) = \frac{F(u^s - 1)}{\eta(u^s - u)}$$

where $\eta(u^s - u) = u^s - u$ or 1, according as $p \mid w_k$ or not. Since Iwasawa main conjecture asserts that $F(T)$ and $f_k^-(T)$ are equal up to an integral invertible power series of $\mathbb{Z}_p[[T]]$, it leads to the following non vanishing of the p-adic L-function. If k is a CM-field which satisfies the Gross conjecture, then

$$\lim_{s\to 0} L_p(\chi\omega, s)/s^{r(k)} \neq 0.$$

3. Norm comparable elements of local and global fields

For a number field k and a finite prime $\mathfrak{p}$ of k, let $k_\mathfrak{p}$ denote the completion of k at $\mathfrak{p}$. Over a $\mathbb{Z}_p$-extension $k_{\mathfrak{p},\infty} = \bigcup_n k_{\mathfrak{p},n}$ of the local field $k_\mathfrak{p}$, local class field theory tells us the isomorphism, $k_\mathfrak{p}^\times / N_n k_{\mathfrak{p},n}^\times \cong G(k_{\mathfrak{p},n}/k_\mathfrak{p})$ where N_n denotes the norm map $N_{k_{\mathfrak{p},n}/k_\mathfrak{p}}$ for $k_{\mathfrak{p},n}/k_\mathfrak{p}$. It follows by taking inverse limits that

$$\varprojlim_n \frac{\overline{k}_\mathfrak{p}}{N_n \overline{k}_{\mathfrak{p},n}} \cong \varprojlim_n G(k_{\mathfrak{p},n}/k_\mathfrak{p}) = G(k_{\mathfrak{p},\infty}/k_\mathfrak{p}) = \mathbb{Z}_p$$

where the p-adic completion $\overline{k}_\mathfrak{p}$ decomposes into $\overline{k}_\mathfrak{p} = \varprojlim_n k_\mathfrak{p}^\times/(k_\mathfrak{p}^\times)^{p^n} \cong \mathbb{Z}_p \oplus U_\mathfrak{p}^1$ for the principal units $U_\mathfrak{p}^1 \subset U_\mathfrak{p}$ of $k_\mathfrak{p}$ which is compact. The surjection $\overline{k}_\mathfrak{p} \to$

$\varprojlim_n \overline{k}_{\mathfrak{p}}/N_n\overline{k}_{\mathfrak{p},n} \to 1$ which follows from the compactness of $\overline{k}_{\mathfrak{p}}$ induces the following exact sequence. $1 \to \bigcap_n (N_n\overline{k}_{\mathfrak{p},n}) \to \overline{k}_{\mathfrak{p}} \to \varprojlim_n \overline{k}_{\mathfrak{p}}/N_n\overline{k}_{\mathfrak{p},n} \to 1$. By a compactness argument of local field, it follows that $\overline{k}_{\mathfrak{p}}^{\text{univ}} = \bigcap_n (N_n\overline{k}_{\mathfrak{p},n}) = \overline{k}_{\mathfrak{p}}^{\text{coh}} = \pi(\varprojlim_n \overline{k}_{\mathfrak{p},n})$ where $\varprojlim_n \overline{k}_{\mathfrak{p},n}$ denotes the inverse limit of $\overline{k}_{\mathfrak{p},n}$ with respect to the norm maps and π denotes the natural projection from $\varprojlim_n \overline{k}_{\mathfrak{p},n}$ to $\overline{k}_{\mathfrak{p}}$.

Lemma 3.1. $\overline{k}_{\mathfrak{p}}/\overline{k}_{\mathfrak{p}}^{\text{univ}} = \overline{k}_{\mathfrak{p}}/\overline{k}_{\mathfrak{p}}^{\text{coh}} \cong G(k_{\mathfrak{p},\infty}/k_{\mathfrak{p}})$.

We use the definition of $k_{\mathfrak{p}}^{\text{univ}} = k_{\mathfrak{p}}^{\text{coh}}$ for the group of universal norm elements the local field $k_{\mathfrak{p}}^{\times}$ for $\mathfrak{p}|p$. Even if the decomposition group is the closure of the image of $k_{\mathfrak{p}}^{\times}$ via Artin map, but the quotient by the kernel of the Artin map already gives same quotient whether we take the p-adic closure or not. Hence, there are following isomorphisms(cf. [**?**])

$$\frac{\overline{k}_{\mathfrak{p}}}{\overline{k}_{\mathfrak{p}}^{\text{coh}}} = \frac{k_{\mathfrak{p}}}{k_{\mathfrak{p}}^{\text{coh}}} \cong G(k_{\mathfrak{p}\infty}/k_{\mathfrak{p}}).$$

Over a $\mathbb{Z}_p$-extension $k_\infty = \bigcup_n k_n$ of k, let $N_{m,n} = N_{k_m/k_n}$ denote the norm map from k_m to k_n and let $N_m = N_{m,0}$ denote the norm map from k_m to the ground field $k_0 = k$. We define the concepts of the norm comparable subgroups B_n^{coh} and the universal norm subgroups B_n^{univ} for any subgroup B_n of the intermediate field k_n as follows.

$$B_n^{\text{coh}} = \pi(\varprojlim_{m\geq n} B_m),\ B_n^{\text{univ}} = \bigcap_{m\geq n} N_{k_m/k_n}B_m$$

where the inverse limits are taken with respect to the norm maps and $\pi = \pi_n$ denotes the natural projection from $\varprojlim_{m\geq n} B_m$ to B_n defined as $\pi((b_m)_{m\geq n}) = b_n$. For instance, $U_n(p)^{\text{coh}} = \pi(\varprojlim_{m\geq n} U_m(p))$, $U_n(p)^{\text{univ}} = \bigcap_{m\geq n} N_{k_m/k_n}U_m(p)$, the universal norm comparable elements under tensor product $(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\text{coh}} = \pi(\varprojlim_{m\geq n}(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p))$ and the universal norm elements under tensor product $(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\text{univ}} = \bigcap_{m\geq n} N_{k_m/k_n}(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)$. Then, π induces the following projection

$$\pi : (\varprojlim(U_n(p) \otimes \mathbb{Z}_p))_\Gamma \to U_k(p) \otimes \mathbb{Z}_p$$

where $(\varprojlim(U_n(p) \otimes \mathbb{Z}_p))_\Gamma$ is the coinvariants of the inverse limit $\varprojlim(U_n(p) \otimes \mathbb{Z}_p)$ by $\Gamma = G(k_\infty/k)$. For the local field k_v which is the completion of k at a finite place v, let $k_{v,\infty}$ be the corresponding $\mathbb{Z}_p$-extension of k_v. By the definition, we have $k_v^{\text{coh}} = \pi(\varprojlim_n k_{v,n}^{\times})$ where $k_{v,n}$ is the subfield of $k_{v,\infty}$ of degree p^n over k_v. Write k^{loc} for the set of all elements which are locally norm coherent,

$$k^{\text{loc}} = \{\alpha \in k^\times \,|\, \text{there is } (\alpha_{v,n}) \in \varprojlim_n k_{v,n} \text{ such that } \alpha_{v,0} = \alpha \text{ for all } v\}.$$

It follows that $k^{\text{loc}} = k^\times \bigcap_v k_v^{\text{coh}}$ for all finite places of k. Then by a well known property of local compactness, we have $\bigcap_n N_n k_{v,n} = k_v^{\text{coh}}$. Since k_∞/k is unramified at primes prime to p and k^{loc} is p-units, it follows that $k^{\text{loc}} = U_k(p) \bigcap_{v|p,n} N_n k_{v,n}$. We defined a filtration of $U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in the following way.

$$U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p \supseteq (U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\text{univ}} \supseteq (U_n(p)^{\text{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\text{coh}} \supseteq U_n(p)^{\text{coh}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

For a number field F, let $\text{Fr}_{\mathbb{Z}_p}(U_F(p) \otimes \mathbb{Z}_p)$ denote the $\mathbb{Z}_p$-torsion free part of the group of p-units of F, i.e., $\text{Fr}_{\mathbb{Z}_p}(U_F(p) \otimes \mathbb{Z}_p) = U_F(p) \otimes \mathbb{Z}_p/\text{Tor}_{\mathbb{Z}_p}(U_F(p) \otimes \mathbb{Z}_p)$. Let

π^{fr} denote the corresponding projection induced from $\pi : (\varprojlim (U_n(p) \otimes \mathbb{Z}_p))_\Gamma \to U_k(p) \otimes \mathbb{Z}_p$

$$\pi^{\mathrm{fr}} : (\varprojlim \mathrm{Fr}_{\mathbb{Z}_p}(U_n(p) \otimes \mathbb{Z}_p))_\Gamma \to \mathrm{Fr}_{\mathbb{Z}_p}(U_k(p) \otimes \mathbb{Z}_p).$$

Then

$$\mathrm{Im}(\pi) = (U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}} \text{ and } \mathrm{Im}(\pi^{\mathrm{fr}}) = (\mathrm{Fr}(U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}.$$

Theorem 3.2. *Let k be a number field with $k_\infty^{\mathrm{cyc}} = \bigcup_n k_n$. Then $T_p(k)^\Gamma = 1$ if and only if $N_n(U_n(p)^{\mathrm{univ}}) = U_k(p)^{\mathrm{univ}}$ for all $n > 0$.*

Proof. We first suppose that $N_n(U_n(p)^{\mathrm{univ}}) = U_k(p)^{\mathrm{univ}}$ for all $n \geq 0$. We start with the following proposition.

Proposition 3.3. *Let k be a number field such that $\bigcap_{n \in \mathbb{N}} N_n(U_n(p)^{\mathrm{univ}}) = U_k(p)^{\mathrm{univ}}$. Then*

$$1 = \frac{U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p}{(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}} = \frac{(\mathrm{Fr}(U_k(p))^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p}{(\mathrm{Fr}(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}}.$$

Proof. We need the following known lemma and we give here a proof for the convenience.

Lemma 3.4. *Let Θ_n be a subgroup of $U_n(p)$ such that $N_{m,n} : \Theta_m \to \Theta_n$. Then we have $(\Theta \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{univ}} = (\Theta \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}$.*

Proof. We use the fact that the p-adic completion $\Theta_n \otimes_{\mathbb{Z}} \mathbb{Z}_p = \varprojlim_r \Theta_n/(\Theta_n)^{p^r}$ of Θ_n is compact. For $r \geq m > 0$ and for $\alpha \in \bigcap_n N_n(\Theta_n \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, let $X_r(\alpha) = N_{r,m}\Theta_r \otimes_{\mathbb{Z}} \mathbb{Z}_p \bigcap N_m^{-1}\alpha$ where $N_m^{-1}\alpha = \{b \in U_m(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p | N_m(b) = \alpha\}$. Since $\alpha \in \bigcap_n N_n(\Theta_n \otimes_{\mathbb{Z}} \mathbb{Z}_p)$, $X_r(\alpha)$ is non empty and compact. The family $X_r(\alpha)$ has the finite intersection property as $r \geq m$ varies because for a finite set of numbers $n_g > ... > n_1 > m$, $X_{n_i}(\alpha)$ is a decreasing chain. It follows that there is $\beta_m \in \bigcap_{r \geq m} N_{r,m}(\Theta_r \otimes_{\mathbb{Z}} \mathbb{Z}_p)$ such that $N_m\beta_m = \alpha$. In this way, one can construct a norm coherent sequence whose first term is α. This completes the proof of lemma 3.4. □

We also have the following lemma.

Lemma 3.5. *$U_k(p)^{\mathrm{loc}} = k^{\mathrm{univ}} = U_k(p)^{\mathrm{univ}}$.*

Proof. It follows immediately from the following Hasse's norm theorem.

Theorem of Hasse. *Let K be a number field and L be a cyclic extension of K. Each $\alpha \in K^\times$ is a norm for L/K if and only if for each places $\mathfrak{p}$ of K and $\mathfrak{P}$ of L lying over $\mathfrak{p}$, α is a norm for $L_{\mathfrak{P}}/K_{\mathfrak{p}}$.* □

Again Lemmas 3.4 and 3.5, we are lead to

$$\pi(\varprojlim_n (U_n(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = (U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}$$
$$= (U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{univ}} \supset ((U_k(p)^{\mathrm{loc}})^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

From the assumption, we have the following equality.

$$(U_k(p)^{\mathrm{loc}})^{\mathrm{univ}} = (U_k(p)^{\mathrm{univ}})^{\mathrm{univ}} = \bigcap_{n=1}^{\infty} N_n(\bigcap_{m \geq n} N_{m,n} U_m(p)) = U_k(p)^{\mathrm{coh}} = U_k(p)^{\mathrm{univ}}.$$

It follows from $(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}} \supset ((U_k(p)^{\mathrm{loc}})^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = U_k(p)^{\mathrm{loc}} \otimes \mathbb{Z}_p$ that

$$1 = \frac{U_k(p)^{\mathrm{loc}} \otimes \mathbb{Z}_p}{(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}}.$$

It follows directly from the definitions that the roots of unity in k are norm comparable both in local and global senses over the cyclotomic $\mathbb{Z}_p$-extension. Hence, we can divide the equation above by the group of roots of unity in order to get an identity on the free parts. We complete the proofs of Proposition 3.2. □

We fix $k_\infty^{\mathrm{cyc}} = \bigcup_n k_n$ the cyclotomic $\mathbb{Z}_p$-extension of k. We write the corresponding maps λ_c, κ_c for this cyclotomic $\mathbb{Z}_p$-extension. Following Kuz'min, we define Tate module $T_p(k)$ as the inverse limit of $G(L_n/k_n)$.

$$T_p(k) = \varprojlim_n G(L_n/k_n).$$

We describe $U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ as the kernel of an exact sequence. This module plays an essential role in the description of the Galois invariants of the Tate module. In order to avoid a confusion $U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ with $(U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{loc}}$, we will give a concrete explanation for the module $U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ which will also be used in the next proposition. We investigate Sinnott's exact sequence in the appendix of [?]. Let L_0 denote the maximal p-extension of k^{cyc} which is abelian over k, unramified at primes outside p and $(k_\infty^{\mathrm{cyc}})_{\mathfrak{p}} = (L_0)_{\mathfrak{P}}$ at primes $\mathfrak{P} \mid \mathfrak{p}$ dividing p. Let H_k be the maximal abelian p-extension of k which is unramified at primes outside p and splits completely at primes over p. There are two essentially same exact sequences from Kuz'min and Sinnott. Let A_k and B_k denote respectively the class groups in the idele group J_k corresponding to L_0 and H_k. Class field theory [?] shows the following exact sequence.

$$1 \longrightarrow \delta_k(U_k(p)) \otimes \mathbb{Z}_p \longrightarrow \prod_{v|p} \frac{k_v^\times}{k_v^{\mathrm{loc}}} \longrightarrow J_k/A_k[p] \longrightarrow J_k/B_k[p] \longrightarrow 1$$

where δ_k denotes the diagonal imbedding and $M[p]$ denotes the p-adic completion for a module M. Using the p-adic logarithm, Sinnott [?] gave the following similar exact sequence.

$$1 \longrightarrow \overline{g_k(U_k(p))} \longrightarrow \overline{g_k(k^\times)} \longrightarrow G(L_0/k) \longrightarrow G(H_k/k) \longrightarrow 1$$

where as defined in §2, g_k denotes the map $g_k : k^\times \to \sum_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v$ and the overline denotes the topological closure. Then it is known([?]) that

$$\overline{g_k(k^\times)} = \sum_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v \cong \prod_{v|p} \frac{k_v^\times}{k_v^{\mathrm{loc}}} \cong \prod_{v|p} \mathbb{Z}_p.$$

Let $\overline{\delta}_k = \delta_k \otimes id$ be the $\mathbb{Z}_p$-linear map

$$\overline{\delta}_k = \delta_k \otimes id : U_k(p) \otimes \mathbb{Z}_p \to \delta_k(U_k(p)) \otimes \mathbb{Z}_p \subset \prod_{v|p} \frac{k_v^\times}{k_v^{\mathrm{loc}}}$$

obtained by extending $\delta_k : U_k(p) \to \delta_k(U_k(p)) \subset \prod_{v|p} k_v^\times/k_v^{\mathrm{loc}}$ by $\mathbb{Z}_p$-linearity. In the same way, let $\overline{g_k}$ denote the extension of g_k obtained by extending g_k by

$\mathbb{Z}_p$-linearity.

$$\overline{g_k} = g_k \otimes id : U_k(p) \otimes \mathbb{Z}_p \to \prod_{v|p} k_v^\times / k_v^{\mathrm{loc}}.$$

Since $\mathbb{Z}_p$ is a flat $\mathbb{Z}$-module and the kernel $\mathrm{Ker}(g_k|_{U_k(p)})$ of $g_k|_{U_k(p)}$ is $U_k(p)^{\mathrm{loc}}$, it follows from the following short exact sequence

$$1 \longrightarrow Ker(g_k|_{U_k(p)}) \longrightarrow U_k(p) \xrightarrow{g_k} \overline{g_k(U_k(p))}$$

that

$$1 \longrightarrow U_k(p)^{\mathrm{loc}} \otimes \mathbb{Z}_p \longrightarrow U_k(p) \otimes \mathbb{Z}_p \xrightarrow{\overline{g_k}} \overline{g_k(U_k(p))}$$

which also shows the kernel of $\overline{\delta}_k$, $\mathrm{Ker}(\overline{\delta}) = \mathrm{Ker}(\overline{g_k}) = \mathrm{Ker}(g_k|_{U_k(p)}) \otimes \mathbb{Z}_p = U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Note that $\overline{g_k}$ induces the map α_k as was explained by Kato in the introduction.

$$\begin{tikzcd}
U_k(p) \otimes \mathbb{Z}_p \arrow[r, "\overline{g_k}"] \arrow[dr] & \sum_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(k_v^\times) \cdot v \arrow[r, "\mathrm{Ar}_k"] & \oplus_{v|p} G(k_\infty^{\mathrm{cyc}}/k)_v \\
& U_k(p) \otimes \mathbb{Z}_p / (U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}} \arrow[ur, "\alpha_k"] &
\end{tikzcd}$$

where $\mathrm{Ar}_k(\sum_{v|p} \log_p N_{k_v/\mathbb{Q}_p}(a_v) \cdot v) = \prod_{v|p}(a_v, L_0/k)$. We summarize this in the following lemma which is essentially Proposition 7.4 of [?] when one replaces g_k by δ_k.

Lemma 3.6.

$$1 \longrightarrow U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \longrightarrow U_k(p) \otimes \mathbb{Z}_p \longrightarrow \overline{g_k(U_k(p))} \longrightarrow 1.$$

The first term in Lemma 3.6 is the denominator of the following proposition. The proposition is due to Kuz'min which was proved by using global class field theory(cf. [?]). We modified the proposition and its proof of Kuz'min for our purpose.

Proposition 3.7. *Let k_∞^{cyc} be the cyclotomic $\mathbb{Z}_p$-extension of k. Then*

$$T_p(k)^\Gamma = \frac{U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p}{(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}}.$$

Proof. Let $\overline{X}$ be the Galois group $G(F/k_\infty^{\mathrm{cyc}})$ of the maximal abelian p-extension F of k_∞^{cyc}. Then $\overline{X}$ is a regular Γ-module by Kuz'min(Lemma 7.4 of Kuz'min [?]). Let also $\overline{W}$ denote the subgroup of $\overline{X}$ generated by the inertia group for $\mathfrak{p} \nmid p$ and the decomposition subgroups for $\mathfrak{p} \mid p$. There exists exact sequence

$$1 = \overline{X}^\Gamma \longrightarrow T_p(k)^\Gamma \longrightarrow \overline{W}_\Gamma \xrightarrow{\beta} \overline{X}_\Gamma \longrightarrow T_p(k)_\Gamma \longrightarrow 1.$$

induced from $1 \longrightarrow \overline{W} \longrightarrow \overline{X} \longrightarrow T_p(k) \longrightarrow 1$. Then $\overline{X}_\Gamma = G(F_0/k_\infty^{\mathrm{cyc}})$ where F_0 denotes the maximal abelian subextension of F over k. If L_0 denotes the maximal subextension of F such that unramified outside $\mathfrak{p} \nmid p$ and $L_{0,\mathfrak{P}} = k_{\infty,\mathfrak{p}}^{\mathrm{cyc}}$ over primes $\mathfrak{P}$ dividing primes $\mathfrak{p}$ of k_∞^{cyc}, then $T_p(k)_\Gamma = G(L_0/k_\infty^{\mathrm{cyc}})$. By class field theory, image of β has the following isomorphism(equation 7.11 of Kuz'min [?]).

$$\mathrm{Im}(\beta) = \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v / (k^\times \bigcap \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v) \otimes \mathbb{Z}_p$$

which by class field theory came from the fact that $\mathrm{Im}(\beta)$ is the closure of the following isomorphism,

$$\prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v k^\times/k^\times = \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v/(k^\times \bigcap \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v)$$

where $\widehat{U}_v$ denotes the norm comparable elements k_v^{coh} of $k_v^\times$. By replacing the ground field k by k_n,

$$\overline{W} = \varprojlim_n \overline{W}_{\Gamma_n} = \varprojlim_n \prod_{v\nmid p} U_{n,v} \prod_{v|p} \widehat{U}_{n,v}/\varprojlim_n (k_n^\times \bigcap \prod_{v\nmid p} U_{n,v} \prod_{v|p} \widehat{U}_{n,v}) \otimes \mathbb{Z}_p$$

Since

$$(k_n^\times \bigcap \prod_{v\nmid p} U_{n,v} \prod_{v|p} \widehat{U}_{n,v}) \otimes \mathbb{Z}_p = U_n(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$$

the expression of $\overline{W}$ above leads to

$$1 \longrightarrow \varprojlim_n (U_n(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p) \longrightarrow \varprojlim_n \prod_{v\nmid p} U_{n,v} \prod_{v|p} \widehat{U}_{n,v} \longrightarrow \overline{W} \longrightarrow 1.$$

The short exact sequence above yields

$$\longrightarrow (\varprojlim_n (U_n(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p))_\Gamma \xrightarrow{\alpha} (\varprojlim_n \prod_{v\nmid p} U_{n,v} \prod_{v\nmid p} \widehat{U}_{n,v})_\Gamma \longrightarrow \overline{W}_\Gamma \longrightarrow 1.$$

Since $\varprojlim_n \prod_{v\nmid p} U_{n,v} \prod_{v\nmid p} \widehat{U}_{n,v}$ is cohomologically trivial by Kuz'min, it reduces to via natural projection

$$(\varprojlim_n \prod_{v\nmid p} U_{n,v} \prod_{v|p} \widehat{U}_{n,v})_\Gamma = \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v.$$

Under the identification of above, image of α corresponds to

$$\mathrm{Im}(\alpha) = \pi(\varprojlim_n (U_n(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = (U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}.$$

Hence

$$\overline{W}_\Gamma = \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v/(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}.$$

Now, image of β reads

$$\mathrm{Im}(\beta) = \overline{W}_\Gamma/\mathrm{Ker}(\beta) = \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v/(k^\times \bigcap \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v) \otimes \mathbb{Z}_p.$$

Hence

$$\begin{aligned} T_p(k)^\Gamma = \mathrm{Ker}(\beta) &= (k^\times \bigcap \prod_{v\nmid p} U_v \prod_{v|p} \widehat{U}_v) \otimes \mathbb{Z}_p/(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}} \\ &= \frac{U(p)_k^{\mathrm{loc}} \otimes \mathbb{Z}_p}{(U_k(p)^{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}}. \end{aligned}$$

We complete the proof of Proposition 3.7. □

Propositions 3.3 and 3.7 implies that $T_p(k)^\Gamma = 0$. Conversely, we suppose that $T_p(k)^\Gamma = 0$. Lemma 3.5 and Theorem 3.2 lead to the following identity.

$$(U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}} = U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

By definition, for each m, the norm map N_m obviously maps $(U_m(p)\otimes_{\mathbb{Z}}\mathbb{Z}_p)^{\mathrm{coh}}$ onto $(U_k(p)\otimes_{\mathbb{Z}}\mathbb{Z}_p)^{\mathrm{coh}}$. Hence, it leads to the following surjection.

$$N_m : U_m(p)^{\mathrm{univ}}\otimes_{\mathbb{Z}}\mathbb{Z}_p \twoheadrightarrow U_k(p)^{\mathrm{univ}}\otimes_{\mathbb{Z}}\mathbb{Z}_p.$$

Hence, it follows from

$$U_n(p)^{\mathrm{univ}}\otimes_{\mathbb{Z}}\mathbb{Z}_p = N_m(U_m(p)^{\mathrm{univ}}\otimes_{\mathbb{Z}}\mathbb{Z}_p) = (N_m(U_m(p)^{\mathrm{univ}})\otimes_{\mathbb{Z}}\mathbb{Z}_p$$

and the flatness of $\mathbb{Z}_p$ that

$$1 = \frac{U_k(p)^{\mathrm{univ}}}{N_mU_m(p)^{\mathrm{univ}}}\otimes_{\mathbb{Z}}\mathbb{Z}_p.$$

However, we have the following inclusion

$$U_m(p)^{\mathrm{univ}}\supset U_k(p)^{\mathrm{univ}}$$

which follows directly by lifting $N_m = \sum_{\sigma\in G(k_m/k)}\sigma$ to $\tilde{N}_m = \sum_{\sigma\in G(k_m/k)}\tilde{\sigma}$ via the lifting for the elements σ of $G(k_m/k)$ into $\tilde{\sigma}$ over any $\varprojlim G(k_{m+i}/k)$. Since

$$U_k(p)^{\mathrm{univ}}\supset N_mU_m(p)^{\mathrm{univ}}\supset (U_k(p)^{\mathrm{univ}})^{p^m}$$

the order $\#(\frac{U_k(p)^{\mathrm{univ}}}{N_m(U_m(p)^{\mathrm{univ}})})$ is of p-power and hence,

$$1 = \frac{U_k(p)^{\mathrm{univ}}}{N_mU_m(p)^{\mathrm{univ}}}\otimes_{\mathbb{Z}}\mathbb{Z}_p = \frac{U_k(p)^{\mathrm{univ}}}{N_mU_m(p)^{\mathrm{univ}}}.$$

By the above equality, $U_k(p)^{\mathrm{univ}} = \bigcap N_mU_m(p)^{\mathrm{univ}}$. This completes the proof of Theorem 3.2. □

Based on Theorem 3.2, we make the following conjecture on the Tate module of an arbitrary number field k.

Conjecture. *For a number field k, the Galois invariants of Tate module is trivial.*

$$T_p(k)^{\Gamma} = 1.$$

Note that this conjecture is stronger than the Gross conjecture. However, the evidence can be found in the following technical point which seems very plausible.

$$T_p(k)^{\Gamma} = 1 \Longleftrightarrow N_{n+1,n}U_{n+1}(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{univ}} \text{ for all } n\geq 0.$$

Examples for this conjecture are given after the next corollary. For any field k, we have the following implication.

$$U_k(p)^{\mathrm{univ}} = U_k(p)^{\mathrm{coh}} \Longrightarrow T_p(k)^{\Gamma} = 1.$$

We will discuss the converse in Proposition 3.11. Over the $\mathbb{Z}_p$-tower $k_\infty^{\mathrm{cyc}} = \bigcup_n k_n$, we have the following equivalent condition.

Corollary 3.8. *Let k be a number field with $k_\infty^{\mathrm{cyc}} = \bigcup_n k_n$. Then*

$$U_n(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{coh}} \textit{ for all } n\geq 0 \Longleftrightarrow T_p(k_n)^{\Gamma_n} = 1 \textit{for all } n\geq 0.$$

Proof. If $U_n(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{coh}}$ for all $n\geq 0$, then it follows that $N_{m,n}U_m(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{univ}}$ for $m\geq n$ by the assumption. Hence, Theorem 3.2 shows that $T_p(k_n)^{\Gamma_n} = 1$ for all $n\geq 0$. Conversely, if $T_p(k_n)^{\Gamma_n} = 1$ for all $n\geq 0$, then $N_{m,n}U_m(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{univ}}$ for $m\geq n$ by Theorem 3.2. By applying the same process successively, we can construct a norm comparable sequence from any element of $U_n(p)^{\mathrm{univ}}$ for all n. □

Examples. We will give some examples of k for which $T_p(k_n)^{\Gamma_n} = 1$ for all $n \geq 0$ over k_∞^{cyc}. Firstly, the most simple case is any field k such that there is only one prime $\mathfrak{p}$ of k lying over p. For such case(cf. [**?**]), we know that $U_n(p) = U_n(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{coh}}$ which implies $T_p(k_n)^{\Gamma_n} = 1$ for all $n \geq 0$. Secondly, we suppose that k is an abelian extension of the rational field. Then k_n contains a group $C_n(p)$ of circular numbers as defined in [**?**] which is a subgroup of $U_n(p)$. The class number formula of Sinnott [**?**] implies the index $(U_n(p) : C_n(p))$ is finite and in many cases $(C_n(p)) : C_n^{\mathrm{coh}}(p))$ is finite where $C_n^{\mathrm{coh}}(p)) = U_n(p)^{\mathrm{univ}} \bigcap C_n(p)$. In special, this is the case when the following quotient is finite

$$(\frac{C_I(p)}{C_I(p) \bigcap C_n^{\mathrm{coh}}(p)}) \otimes \mathbb{Z}_p < \infty$$

where $C_I(p)$ is the group of circular numbers of the decomposition field at p(cf. [**?**]) over k_∞^{cyc}. We leave the explicit examples which are easy to find to the reader. Hence, such examples by the class number formula of Sinnott satisfy the finite condition

$$(U_n(p) : U_n(p)^{\mathrm{coh}}) < \infty.$$

Since for each n, we can construct $x(n) \geq n$ such that $N_{x(n)/n} U_{x(n)}(p) = U_n(p)$, we can obtain a norm comparable sequence for each universal norm element. Therefore, we have the following equalities.

$$U_n(p) = U_n(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{coh}}$$

together with $T_p(k_n)^{\Gamma_n} = 1$ for all $n \geq 0$.

For the Gross conjecture, we have the following equivalent form.

Proposition 3.9. *Let k be a number field with $k_\infty^{\mathrm{cyc}} = \bigcup_n k_n$. Then the Gross conjecture is true for k if and only if there is $n = n(k) \geq 0$ such that*

$$N_{n+i} U_{n+i}(p)^{\mathrm{univ}} = N_n U_n(p)^{\mathrm{univ}} \textit{ for all } i \geq 0.$$

Proof. It follows from Proposition 3.7 that $T_p(k)^\Gamma$ is finite if and only if $(U_k(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p : (N_n U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p)^{\mathrm{coh}}) < \infty$. We know from Lemma 3.4 that $(U_k(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p)^{\mathrm{coh}} = (U_k(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p)^{\mathrm{univ}} = \bigcap_n N_n(U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p) = \bigcap_n (N_n U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p)$ and that

$$N_n U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p \supset (U_k(p)^{\mathrm{univ}})^{p^n} \otimes \mathbb{Z}_p.$$

Hence we have that $T_p(k)^\Gamma$ is finite if and only if the decreasing chain of modules $\{N_n U_n(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p\}$ must stop, that is, there is $n = n(k)$ such that $N_s U_s(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p = N_m U_m(p)^{\mathrm{univ}} \otimes \mathbb{Z}_p$ for $m \geq s \geq n$. Since the index $(N_s U_s(p)^{\mathrm{univ}} : N_m U_m(p)^{\mathrm{univ}})$ is a p-primary, the condition leads to

$$N_s U_s(p)^{\mathrm{univ}} = N_m U_m(p)^{\mathrm{univ}} \text{ for } m \geq s \geq n.$$

□

Since for all n, $N_n U_n(p)^{\mathrm{univ}}$ lies between $U_k(p)^{\mathrm{univ}}$ and $U_k(p)^{\mathrm{coh}}$, Proposition 3.9 leads to the following corollary.

Corollary 3.10. *If $(U_k(p)^{\mathrm{univ}} : U_k(p)^{\mathrm{coh}}) < \infty$ then the Gross conjecture is true for k.*

We are ready to describe a necessary and sufficient condition for the converse for the following implication. $U_k(p)^{\mathrm{univ}} = U_k(p)^{\mathrm{coh}} \Longrightarrow T_p(k)^{\Gamma} = 1$.

Proposition 3.11. *Suppose the Gross conjecture is true for the intermediate fields k_n of k_∞^{cyc} over a number field k. Then*

$$U_k(p)^{\mathrm{univ}} = U_k(p)^{\mathrm{coh}} \Longleftrightarrow T_p(k)^{\Gamma} = 1.$$

Proof. It is enough to show that if $T_p(k)^{\Gamma} = 1$ then $U_k(p)^{\mathrm{univ}} = U_k(p)^{\mathrm{coh}}$. We need the following lemma which will also be used later.

Lemma 3.12. *Under the assumption of the proposition above, we have*

$$(U_k^{\mathrm{univ}})^{\mathrm{univ}} = \bigcap_n N_n U_n(p)^{\mathrm{univ}} = U_k(p)^{\mathrm{coh}}.$$

Proof. By the assumption, there is a function $x : \mathbb{N} \cup \{0\} \to \mathbb{N}$ such that for each $n \in \mathbb{N} \cup \{0\}$, $x(n) > n$ and

$$N_{x^{r+1}(n),x^r(n)} U^{\mathrm{univ}}_{x^{r+1}(n)} = N_{x^{r+1}(n),x^r(n)} N_{x^{r+2}(n),x^{r+1}(n)} U^{\mathrm{univ}}_{x^{r+2}(n)}$$

where $x^r = x \circ \cdots \circ x$ denotes the rth composite of x with $x^0(n) = n$. Let $\alpha \in (U_k^{\mathrm{univ}})^{\mathrm{univ}}$. Then by taking $n = 0$ above, we have $\alpha \in N_{x(0),0} U^{\mathrm{univ}}_{x(0)} = N_{x(0)} U^{\mathrm{univ}}_{x(0)}$ and for each $\alpha_n \in N_{x^{n+1}(0),x^n(0)} U^{\mathrm{univ}}_{x^{n+1}(0)}$, we can find $\alpha_{n+1} \in N_{x^{n+2}(0),x^{n+1}(0)} U^{\mathrm{univ}}_{x^{n+2}(0)}$ such that

$$\alpha_n = N_{x^{n+1}(0),x^n(0)} \alpha_{n+1}.$$

This gives rise to a norm comparable sequence $\{\alpha_n\}_{n \in \mathbb{N} \cup \{0\}}$ with $\alpha_0 = \alpha$ as we claimed. $\square$

The proposition follows from Lemma 3.12 and Theorem 3.2. $\square$

Proposition 3.13. *The following conditions are equivalent.*

$$(U_n^{\mathrm{univ}} : U_n^{\mathrm{coh}}) < \infty, \textit{ for all } n \geq 0 \iff \textit{The Gross conjecture is true for all } k_n$$

Proof. If $(U_n^{\mathrm{univ}} : U_n^{\mathrm{coh}}) < \infty$ for all n, then each k_n satisfies the Gross conjecture by Corollary 3.10. Conversely, if each k_n satisfies the Gross conjecture then it follows from Proposition 3.9 that

$$(U_n^{\mathrm{univ}} : (U_n^{\mathrm{univ}})^{\mathrm{univ}}) < \infty.$$

It follows from Lemma 3.12 that $(U_n^{\mathrm{univ}} : U_n^{\mathrm{coh}}) < \infty$ for all n. This completes the proof. $\square$

The following corollary shows that the coh-functor commutes with the tensor product $\otimes_{\mathbb{Z}} \mathbb{Z}_p$ when the intersection commutes with the tensor product $\otimes_{\mathbb{Z}} \mathbb{Z}_p$. Notice that the first examples mostly satisfy this condition as well.

Corollary 3.14. *If $\bigcap_n N_n(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p) = (\bigcap_n N_n U_n(p)) \otimes_{\mathbb{Z}} \mathbb{Z}_p$, then it follows that*

$$(U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}} = U_k(p)^{\mathrm{coh}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Proof. We know by the compactness argument that the following identity holds.

$$(U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}} = \bigcap_n N_n(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p).$$

By the assumption, it follows that $\bigcap_{n=1}^{\infty}(N_n(U_n(p)) \otimes_{\mathbb{Z}} \mathbb{Z}_p) = (\bigcap_{n=1}^{\infty} N_n U_n(p)) \otimes_{\mathbb{Z}} \mathbb{Z}_p = U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. By definition, the norm map $N_{m,n}$ obviously maps $(U_m(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}$ onto $(U_n(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}}$. Hence, it leads to the following surjection.

$$N_{m,n} : U_m(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \twoheadrightarrow U_n(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

Hence, it follows from $U_n(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = N_{m,n} U_m(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = N_{m,n} U_m(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$ that

$$1 = \frac{U_n(p)^{\mathrm{univ}}}{N_{m,n} U_m(p)^{\mathrm{univ}}} \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

However, we know the following inclusion $U_m(p)^{\mathrm{univ}} \supset U_n(p)^{\mathrm{univ}}$. Since

$$U_n(p)^{\mathrm{univ}} \supset N_{m,n} U_m(p)^{\mathrm{univ}} \supset (U_n(p)^{\mathrm{univ}})^{p^{m-n}}$$

it follows that

$$U_n(p)^{\mathrm{univ}} = N_{m,n} U_m(p)^{\mathrm{univ}}.$$

By applying the above equality, $U_n(p)^{\mathrm{univ}} = N_{m,n} U_m(p)^{\mathrm{univ}}$, for all successive fairs $m \geq n$, we can construct a norm comparable sequence from an universal norm element of $U_k(p)^{\mathrm{univ}}$. It leads to $U_n(p)^{\mathrm{univ}} = U_n(p)^{\mathrm{coh}}$ and hence $(U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{coh}} = (U_k(p) \otimes_{\mathbb{Z}} \mathbb{Z}_p)^{\mathrm{univ}} = U_k(p)^{\mathrm{univ}} \otimes_{\mathbb{Z}} \mathbb{Z}_p = U_k(p)^{\mathrm{coh}} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. This completes the proof. $\square$

Remark. Under the assumption of Corollary 3.14, we can compute the quotient of $U_n(p)/U_n(p)^{\mathrm{coh}}$ using results of Kuz'min. Let $U_n'(p) = U_n(p)/\mu_n$ and let

$$U_\infty' = \varprojlim U_n'(p)$$

where the inverse limit is taken with respect to the norm maps. We need the following Theorems of Kuz'min.

Theorem 7.2 of [?]. *Let k be a number field and let r_1 and r_2 be the number of real and complex places of k. Then U_∞' is a free Γ-module of rank $r_1 + r_2$.*

Theorem 7.3 of [?]. *Let k be a number field. Then*

$$(U_n'(p) \otimes \mathbb{Z}_p)^{\mathrm{coh}} \cong (U_\infty')_\Gamma.$$

Notice that by replacing k with k_n, the above theorems tells us that $(U_n'(p) \otimes \mathbb{Z}_p)^{\mathrm{coh}}$ is a free $\mathbb{Z}_p[G(k_n/k)]$-module of $r_1 + r_2$. Corollary 3.14 leads to the following isomorphism.

$$\frac{U_n(p)}{U_n(p)^{\mathrm{coh}}} \otimes \mathbb{Z}_p = \frac{U_n(p) \otimes \mathbb{Z}_p}{(U_n(p) \otimes \mathbb{Z}_p)^{\mathrm{coh}}} = \frac{U_n'(p) \otimes \mathbb{Z}_p}{(U_n'(p) \otimes \mathbb{Z}_p)^{\mathrm{coh}}} \cong \mathbb{Z}_p^{r_n - 1}$$

where r_n denotes the number of primes of k_n dividing p. Hence, it follows that

$$\mathrm{rk}_{\mathbb{Z}} \frac{U_n(p)}{U_n(p)^{\mathrm{coh}}} = r_n - 1.$$

Notice that this $\mathbb{Z}$-rank $r_n - 1$ must be stabilized at a certain level.

As we mentioned in the introduction, we can guess the following isomorphism on the Structure of the Galois invariants of the Tate module of a number field. For a number field k, there exists an isomorphism of abelian groups.

$$T_p(k)^\Gamma \cong \frac{U_k(p)^{\mathrm{univ}}}{U_k(p)^{\mathrm{coh}}}.$$

The following proposition shows that the Gross conjecture implies the isomorphism above.

Proposition 3.15. *Let k be a number field such that $\{k_n\}_n$ satisfy the Gross conjecture. Then we have an isomorphism of abelian groups.*

$$T_p(k)^\Gamma \cong \frac{U_k(p)^{\mathrm{univ}}}{U_k(p)^{\mathrm{coh}}}.$$

Proof. What we will prove is that the Gross conjecture for all intermediate fields k_n with $n \in \{0\} \cup \mathbb{N}$ of k_∞^{cyc} implies the guess on the structure for k_n of k_∞^{cyc}. Theorem 3.2, Lemmas 3.4, 3.5 and Proposition 3.7 imply that the isomorphism of

$$T_n(k)^\Gamma \cong \frac{U_n(p)^{\mathrm{univ}}}{U_n(p)^{\mathrm{coh}}} \otimes \mathbb{Z}_p$$

reduces to the following triviality.

$$\frac{(U_n(p)^{\mathrm{univ}})^{\mathrm{univ}}}{U_n(p)^{\mathrm{coh}}} \otimes \mathbb{Z}_p = 1$$

which follows from Lemma 3.12 by replacing k by k_n. By Lemma 3.12 and the following filtration

$$(U_n(p)^{\mathrm{univ}})^{p^{m-n}} \subset N_{m,n}U_m(p)^{\mathrm{univ}} \subset U_n(p)^{\mathrm{univ}}$$

it follows that the quotient group $U_n(p)^{\mathrm{univ}}/U_n(p)^{\mathrm{coh}}$ is a p-primary group. This completes the proof. □

More precisely, we have the following evidence. RHS of the guess is trivial for the examples after Corollary 3.8 and hence Corollary 3.8 shows that the guess above is true for such examples. It follows in general from the remark after Corollary 3.10 that if RHS is trivial then so is LHS and Corollary 3.10 shows that if RHS is finite then so is LHS.

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