# Vector Bundles over Normal Varieties Trivialized by Finite Morphisms

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**Abstract.** Let $Y$ be a normal and projective variety over an algebraically closed field $k$ and $V$ a vector bundle over $Y$. We prove that if there exist a $k$-scheme $X$ and a finite surjective morphism $g : X \to Y$ that trivializes $V$ then $V$ is essentially finite.

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## Contents



## 1 Introduction

Essentially finite vector bundles over a reduced, connected and proper scheme $Y$ over a perfect field $k$ have been defined by Nori in [5] and [6]. They turn out to be those vector bundles $V$ over $Y$ which are trivialized by some principal $G$ bundle $f : Z \to Y$ for a certain finite $k$-group scheme $G$ (i.e. $f^*(V)$ is trivial). The aim of this paper is to prove the following

**Theorem 1.1.** *(Cf. Theorem 2.1). Let $k$ be any algebraically closed field and $Y$ a projective and normal variety over $k$. Assume there exist a projective variety $X$ over $k$ and a finite surjective morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V$ is essentially finite.*

When $Y$ is smooth then Theorem 1.1 is well known: it has first been proved by Parameswaran and Subramanian in [7], §3 for $dim(Y) = 1$ provided $g$ is separable. Then it has been subsequently proved by Balaji and Parameswaran for $Y$ smooth and projective of any dimension in [1], §6 provided $g$ is separable.

Then finally in [2] Biswas and Dos Santos have given a different proof: for any finite and surjective $g : X \to Y$, with $Y$ smooth and projective over $k$, they first explain how to reduce to the case of curves ([2], §3) by means of the Grothendieck-Lefschetz theorem for the $S$-fundamental group scheme, then in loc. cit. §4.2 they prove Theorem 1.1 for $Y$ a smooth and projective curve and $g : X \to Y$ separable (the crucial point) and finally they prove the result for any $g$ (loc. cit. §4.3) and $Y$ a smooth and projective curve.

Our proof of Theorem 1.1 not only holds for $Y$ normal but it is shorter and we use neither Tannakian categories nor Grothendieck-Lefschetz theorem for the fundamental group scheme. The main argument of our proof is Lemma 2.5, which is of independent interest, where we prove that a finite and surjective morphism $g : X \to Y$ between normal and projective varieties over any algebraically closed field $k$, étale outside a closed set of codimension 2 in $Y$, factors through a Galois étale cover $g' : X' \to Y$ if and only if there exists a non trivial vector bundle $V$ on $Y$ such that $g^*(V)$ is trivial on $X$.

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## 2 The theorem

Throughout the whole paper $k$ will be an algebraically closed field and $Y$ a normal and projective variety over $k$. Let us denote by $EF(Y)$ the neutral tannakian category of essentially finite vector bundles over $Y$. The aim of this paper is to prove the following

**Theorem 2.1.** *Assume there exist a normal projective variety $X$ over $k$ and a finite surjective morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

It is clear that Theorem 1.1 can be easily deduced from Theorem 2.1 simply normalizing an irreducible component of $X_{\text{red}}$ dominating $Y$.

**Remark 2.2.** *This theorem holds in both zero and positive characteristic.*

As pointed out in the introduction, the crucial point in the proof of Theorem 2.1 is to prove the statement for $g$ separable (or generically étale, i.e. the extension $K(Y) \subset K(X)$ of the function fields induced by $g$ is separable) and this will be the object of Lemma 2.4.

So first we consider the easier case where $g : X \to Y$ is purely inseparable (i.e. the extension $K(Y) \subset K(X)$ of their function fields is purely inseparable, which only occurs when $char(k) > 0$) and then it will only remain to explain how to reduce to these two cases, the separable and purely inseparable ones.

**Lemma 2.3.** *Assume there exist a normal projective variety $X$ over $k$ and a finite, surjective, purely inseparable morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

*Proof.* We are in the case $char(k) = p > 0$. So let us denote by $F_X : X \to X$ and $F_Y : Y \to Y$ respectively the absolute Frobenius morphisms of $X$ and $Y$. Since $K(Y) \subset K(X)$ is purely inseparable then there exists a positive integer $n$ such that $K(X)^{(p^n)} \subset K(Y)$. This implies that there is a morphism $h : Y \to X$ such that $gh = F_Y^n$ (i.e. the Frobenius iterated $n$ times) and $hg = F_X^n$. By assumption $g^*(V)$ is trivial on $X$, thus $h^*g^*(V) = (gh)^*(V) = (F_Y^n)^*(V)$ is trivial hence $V$ is essentially finite (cf. [4], §2). □

**Lemma 2.4.** *Assume there exist a normal projective variety $X$ over $k$ and a finite, surjective, separable morphism $g : X \to Y$ such that $g^*(V)$ is trivial, then $V \in EF(Y)$.*

*Proof.* We may assume that $K(X)$ is normal (then Galois) over $K(Y)$ with Galois group $G$ (if it is not simply consider the normal closure of the extension $K(Y) \subset K(X)$). Let $W := (g_*\mathcal{O}_X)_{max}$ be the maximal semistable subsheaf of $g_*\mathcal{O}_X$ (i.e. the first term of the Harder-Narasimhan filtration of $g_*\mathcal{O}_X$, [1], §6) then its slope $\mu(W) = \mu_{max}(g_*\mathcal{O}_X) = 0$: indeed since there is at least the canonical morphism $\mathcal{O}_Y \to g_*\mathcal{O}_X$ then in particular we have

$$0 = \mu(\mathcal{O}_Y) \leq \mu_{max}(g_*\mathcal{O}_X);$$

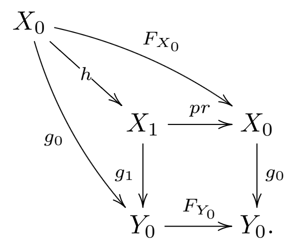
but $g$ is separable then $g^*(W)$ is still semistable; now consider the isomorphism

$$Hom_X(g^*(W), \mathcal{O}_X) \simeq Hom_Y(W, g_*\mathcal{O}_X) \neq 0$$

from which we deduce $\mu(g^*(W)) \leq 0$ hence $\mu(W) \leq 0$ (recall that $\mu(W) = \mu(g^*(W))/deg(g)$).

The coherent sheaf $W$ is in general only torsion free over $Y$. But it is locally free if restricted to a big open subset $Y_0 \subset Y$, i.e. $codim_Y(Y \backslash Y_0) \geq 2$. Let $W_0 := W_{|Y_0}$ denote the vector bundle over $Y_0$, $Sym^*(W_0^*)$ the symmetric algebra of the dual of $W_0$ and consider $X_0 := \mathbf{Spec}(Sym^*(W_0^*))$ with its canonical map $g_0 : X_0 \to Y_0$.

The vector bundle $W_0$ is strongly semistable of degree 0 over $Y_0$: indeed let us denote by $F_{X_0}$ and $F_{Y_0}$ respectively the absolute Frobenius morphisms of $X_0$ and $Y_0$ and assume $W$ is not strongly semistable then there exists a subsheaf $U$ of $F_{Y_0}^*(W)$ such that $deg(U) > 0$. Let $X_1$ be the fiber product of $g_0 : X_0 \to Y_0$ and $F_{Y_0}$. It is an integral scheme. We denote by $pr : X_1 \to X_0$ and $g_1 : X_1 \to Y_0$ the projections and also $h : X_0 \to X_1$ the map given by the universal property of the fiber product:




Now $U \subseteq F_{Y_0}^*(g_{0*}(\mathcal{O}_{X_0})) = g_{1*}(pr^*(\mathcal{O}_{X_0})) = g_{1*}(\mathcal{O}_{X_1})$. But from $\mathcal{O}_{X_1} \hookrightarrow h_*(\mathcal{O}_{X_0})$ we obtain $g_{1*}(\mathcal{O}_{X_1}) \hookrightarrow g_{1*}(h_*(\mathcal{O}_{X_0})) = g_{0*}(\mathcal{O}_{X_0})$ the latter being semistable whence a contradiction. As a consequence we have a homomorphism of $\mathcal{O}_{Y_0}$-algebras $g_{0*}(\mathcal{O}_{X_0}) \simeq W_0$ (cf. also [1], §6).

Since $g_{0*}(\mathcal{O}_{X_0})$ is semistable of slope 0 over $Y_0$ then $X_0$ is a Galois-étale cover over $Y_0$ ([1], Lemma 6.2), the Galois group of $g_0$ still being $G$. Now let us fix some notations: recall that by assumption $V$ is a vector bundle over $Y$ such that $T := g^*(V)$ is trivial on $X$; we set $V_0 := V_{|Y_0}$ and $T_0 := g_0^*(V_0)$ so the latter is also trivial on $X_0$. Since $g_0$ is a Galois-étale cover then $T_0$ is a $G$-bundle on $X_0$. But $X_0$ is a big open set in $X$ thus $G$ acts on $X$ and then $G$ acts also on $T$. Since $T$ is a $G$-bundle then we go on as follows: we have $X/G \simeq Y$ and the trivial bundle $T$ on $X$ descends to $Y$. So by Kempf's lemma (cf. for example [3], Théorème 2.3), for all $x$ in $X$, the stabilizer $G_x$ acts trivially on the fibre $T_x$. But $T$ is trivial and both $X$ and $X_0$ have no global sections except constants, this means that there is a map

$$\rho : G \to GL(T_x) = GL_r$$

over $X$, where $r := rank(T)$. Assume first that the map $\rho : G \to GL_r$ is injective. We already know that $G$ acts freely on $X_0$. So let us take $x \in X \backslash X_0$: since $G_x$ is a subgroup of $G$ then $G_x$ has to be trivial. This proves that $G$ acts freely on $X$. So $g : X \to Y$ is a Galois-étale cover hence $V$ is in $EF(Y)$. Up to now we have assumed $\rho$ to be injective. If it is not then just consider $H := G/ker(\rho)$ and $X' = X/(ker(\rho))$, which is provided with a faithfull $H$-action and clearly $Y \simeq X'/H$. Hence $H \to GL_r$ is injective, $V$ is trivial over $X'$ and we proceed as before.

□

From previous discussion follows Lemma 2.5, which is of independent interest:

**Lemma 2.5.** *Let $Y$ and $X$ be normal and projective varieties and $f : X \to Y$ a finite and surjective morphism, étale outside a closed set of codimension 2 in $Y$, then we have proved that there exists a non trivial vector bundle $V$ on $Y$ such that $f^*V$ is trivial on $X$ if and only if $f$ factors through a Galois étale cover $f' : X' \to Y$.*

*Proof.* The Galois étale cover $f' : X' \to Y$ is the one constructed in the proof of Lemma 2.4. □

**Remark 2.6.** *Let notations be as in Lemma 2.4 and its proof. In Lemma 2.5 we obtain the smallest Galois étale cover where $V$ becomes trivial. Indeed $f' : X' \to Y$ determines and is determined by the kernel of $\rho : G \to GL_r$; if $\rho$ is injective then $X' = X$. If $\rho$ is not injective then $X' := X/ker(\rho)$ is Galois étale over $Y$. It can happen that there are no Galois étale covers between $X$ and $X'$. This happens if and only if $\mu_{max}C < 0$ , where $C$ is the cokernel of $f'_*(\mathcal{O}_{X'}) \to f_*(\mathcal{O}_X)$.*

We are now ready to prove the principal result:

*Proof. of Theorem* 2.1: if $char(k) = 0$ then Lemma 2.4 is sufficient to conclude. So let us assume $char(k) = p > 0$: if $g$ is purely inseparable then Lemma 2.3 is enough to conclude. Otherwise, if $g$ is arbitrary, we argue as follows: again we may assume that $K(X)$ is normal over $K(Y)$ with Galois group $G$. It is known that $L := K(X)^G$ is a proper purely inseparable field extension of $K(Y)$ while $K(X)$ is separable over $L$, then Galois. Let $Z$ be the integral closure of $Y$ in $L$, then $g : X \to Y$ factors through the maps $s : X \to Z$ and $t : Z \to Y$ (i.e. $ts = g$) where $t : Z \to Y$ is purely inseparable and $s : X \to Z$ is separable. By Lemma 2.4 the vector bundle $W := t^*(V) \in EF(Z)$ because $s^*(W)$ is trivial on $X$. As we did for Lemma 2.3, there exists a morphism $h : Y \to Z$ such that $h^*t^*(V) = (th)^*(V) = (F_Y^n)^*(V)$ for some integer $n$; but $h^*(W) \in EF(Y)$ thus $(F_Y^n)^*(V) \in EF(Y)$ then there exists $m \geq n$ such that $(F_Y^m)^*(V)$ is Galois-étale trivial (i.e. there exists a Galois-étale cover $j : Y' \to Y$ such that $j^*((F_Y^m)^*(V))$ is trivial on $Y'$) and that is enough to conclude that $V$ is essentially finite on $Y$. □

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