Extension of Finite Solvable Torsors over a Curve

Marco Antei

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Abstract. Let R be a complete discrete valuation ring with fraction field K and with algebraically closed residue field of positive characteristic p. Let X be a smooth fibered surface over R. Let G be a finite, étale and solvable K-group scheme and assume that either $|G| = p^n$ or G has a normal series of length 2. We prove that for every connected and pointed G-torsor Y over the generic fibre X_η of X there exist a regular fibered surface $\widetilde{X}$ over R and a model map $\widetilde{X} \to X$ such that Y can be extended to a torsor over $\widetilde{X}$ possibly after extending scalars.

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1 Introduction

1.1 Aim and scope

Let S be a connected Dedekind scheme of dimension one and $\eta = Spec(K)$ its generic point; let X be a scheme, $f : X \to S$ a faithfully flat morphism of finite type and $f_\eta : X_\eta \to \eta$ its generic fiber. Assume we are given a finite K-group scheme G and a G-torsor $Y \to X_\eta$. The problem of extending a torsor $Y \to X_\eta$

consists of searching a finite and flat S-group scheme G' whose generic fibre is isomorphic to G and a G'-torsor $T \to X$ whose generic fibre is isomorphic to $Y \to X_\eta$ as a G-torsor. Some solutions to this problems are known in some particular relevant cases, that we briefly recall hereafter. The first important answer to this problem is due to Grothendieck: he proves that, possibly after extending scalars, the problem has a solution when S is the spectrum of a complete discrete valuation ring with algebraically closed residue field of positive characteristic p, with X proper and smooth over S with geometrically connected fibres and $(|G|, p) = 1$ ([11], Exposé X, or [22], Theorem 5.7.10). When S is the spectrum of a discrete valuation ring of residue characteristic p, X is a proper and smooth curve over S then Raynaud suggests a solution, possibly after extending scalars, for G commutative of order a power of p ([19] §3). A similar problem has been studied by Saïdi in [20], §2.4 for formal curves of finite type and $G = (\mathbb{Z}/p\mathbb{Z})_K$. When S is the spectrum of a d.v.r. R of mixed characteristic $(0, p)$ Tossici provides a solution, possibly after extending scalars, for G commutative when X is a regular scheme, faithfully flat over S, with integral fibres provided that the normalization of X in Y has reduced special fibre ([23], Corollary 4.2.8). Finally in [3], §3.3 we provide a solution for G commutative, when S is a connected Dedekind scheme and $f : X \to S$ is a relative smooth curve with geometrically integral fibres endowed with a section $x : S \to X$ provided that Y is pointed over x_η (or, in higher dimension, $f : X \to S$ is a smooth morphism satisfying additional assumptions, cf. [3], §3.2). We stress that in this last case we do not need to extend scalars.

In this paper we study the problem of extending the G-torsor $Y \to X_\eta$ when G is finite, étale and solvable. In general we are not able to extend it over X but we first need to modify X in order to find a solution. More precisely the aim of this paper is to prove the following:

Theorem 1.1. *(Theorem 3.27, Corollary 3.28) Let R be a complete discrete valuation ring with fraction field K and with algebraically closed residue field of characteristic $p > 0$. Let X be a smooth fibered surface over R. Let G be a finite, étale and solvable K-group scheme. We prove that for every connected and pointed G-torsor Y over the generic fibre X_η of X there exist a regular fibered surface $\widetilde{X}$ over R and a model map $\widetilde{X} \to X$ such that Y can be extended to a torsor over $\widetilde{X}$ possibly after extending scalars in the following two cases:*

1. *$|G| = p^n$;*
2. *G has a normal series of length 2.*

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1.2 Notations and conventions

Let S be a connected Dedekind scheme (thus including spectra of fields and discrete valuation rings) and X a S-scheme. We will consider in this article torsors over X under affine and flat S-group schemes G. They are affine, faithfully flat and G-invariant morphisms $p : Y \to X$, locally trivial for the fpqc topology (i.e. $Y \times_X Y \simeq Y \times G$), where Y is a S-scheme endowed with a right action of G and X is endowed with the trivial action. We say that a G-torsor is finite if G is finite and flat. Likewise we say that a G-torsor is commutative (resp. solvable) if G is commutative (resp. solvable). We say that a G-torsor $p : Y \to X$ is pointed if there exists a section $y \in Y(S)$.

When S has dimension one we will denote by $\eta := Spec(K)$ the generic point of S. For any S-scheme T we will denote by T_η its generic fibre $T \times_S Spec(K)$. Thus T_η will always stand for the generic fibre of a scheme T, and not just a scheme over $Spec(K)$. In a similar way for any S-morphism of schemes $v : T \to T'$ we will denote by $v_\eta : T_\eta \to T'_\eta$ the reduction of v over $Spec(K)$. When v_η is an isomorphism we will often say that v is a model map. Let Z be any K-scheme. Any S-scheme T whose generic fibre T_η is isomorphic to Z will be called a model (or S-model) of Z. When Z has an additional structure of group scheme, a model T will be tacitly assumed to have a S-group scheme structure extending the given one on Z. A model map between models of group scheme will be tacitly assumed to be a S-group scheme morphism.

Now let X be a S-scheme, G a finite K-group scheme and $Y \to X_\eta$ a G-torsor. As mentioned in section 1.1 we say that $Y \to X_\eta$ can be extended to X if there exist a finite and flat S-model G' of G and a G'-torsor $T \to X$ whose generic fibre is isomorphic to $Y \to X_\eta$ as a G-torsor.

When $S = Spec(R)$, with R a discrete valuation ring, then $s := Spec(k)$ will always denote the special point of S. For any R-scheme T we will denote by T_s its special fibre $T \times_{Spec(R)} Spec(k)$. Again for any S-morphism of schemes $v : T \to T'$ we will denote by $v_s : T_s \to T'_s$ the reduction of v over $Spec(k)$.

2 Towers of torsors and solvable torsors

2.1 Solvable torsors

Let S be any connected Dedekind scheme, recall that a finite and flat S-group scheme G is said to be solvable if it has a normal series (or solvable series)

$$0 = H_n \lhd H_{n-1} \lhd .. \lhd H_1 \lhd H_0 = G \tag{1}$$

where each H_i is a finite and flat S-group scheme and each quotient H_i/H_{i+1} exists as a S-group scheme and is finite, flat and commutative ($i = 0, .., n-1$). As usual n is called the length of such a normal series.

Remark 2.1. When the H_i are finite and flat then each H_i/H_{i+1} exists and is a finite and flat S-group scheme ([21], §3, Theorem).

Let G be as in (1) then a G-torsor $Y \to X$ can be seen as a tower of commutative torsors, each of them being a H_i/H_{i+1}-torsor: they are called the commutative components of the solvable G-torsor. If for instance $n = 2$ consider the contracted product $Y' := Y \times^G G/H_1$ ([8], III, §4, n° 3) in order to factor Y into a tower of two commutative torsors: a commutative G/H_1-torsor $Y' \to X$ and a commutative H_1-torsor $Y \to Y'$:

$$\begin{tikzcd} Y \arrow[rr, "H_1"] \arrow[dr, "G"'] & & Y' \arrow[dl, "G/H_1"] \\ & X & \end{tikzcd}$$

If $n > 2$ we iterate the process factoring $Y \to Y'$ and so on.

2.2 Towers of torsors

Let S be any connected Dedekind scheme of dimension one, X a scheme and $f : X \to S$ a faithfully flat morphism of finite type. We first consider the following general situation: we are given a finite K-group scheme G (here G is not necessarily solvable) and a G-torsor $Y \to X_\eta$ pointed in $y \in Y(K)$; let G_2 be a non trivial (but not necessarily commutative) normal K-subgroup scheme of G and $G_1 := G/G_2$; we can see $Y \to X_\eta$ as a tower of two torsors: a G_2-torsor $Y_2 = Y \to Y_1$ and a G_1-torsor $Y_1 \to X_\eta$ pointed in $y_1 \in Y_1(K)$, image of y. In Theorem 2.3 we prove that $Y \to X_\eta$ can be extended if and only if both $Y_1 \to X_\eta$ and $Y_2 \to Y_1$ can be extended. We will need Lemma 2.2 which has been pointed out to the author by Marco Garuti. A similar statement with a similar proof (but with different assumptions) can be found in [17], Lemma 1, so we only sketch the proof.

Lemma 2.2. *Let T_1 and T_2 be (resp.) a G_1-torsor over X pointed in $t_1 \in T_1(S)$ and a G_2-torsor over X pointed in $t_2 \in T_2(S)$. Let $\varphi : T_1 \to T_2$ be a X-morphism sending $t_1 \mapsto t_2$, then there is a unique homomorphism $\rho : G_1 \to G_2$ such that φ commutes with the actions of G_1 and G_2.*

Proof. Let $x : S \to X$ be the image of t_1 (and t_2) in $X(S)$. Thus pulling back φ over x we get a morphism $\varphi_x : T_{1x} \to T_{2x}$. But by assumptions we have isomorphisms $\mu_{G_1} : G_1 \to T_{1x}, g \mapsto g \cdot x$ and $\mu_{G_2} : G_2 \to T_{2x}, h \mapsto h \cdot y$. Then we set $\rho := \mu_{G_2}^{-1} \circ \varphi_x \circ \mu_{G_1}$. Now from

$$\begin{tikzcd} T_1 \times G_1 \arrow[r] \arrow[d, "\varphi\times\rho"'] & T_1 \times_X T_1 \arrow[d, "\varphi\times\varphi"] \\ T_2 \times G_2 \arrow[r] & T_2 \times_X T_2 \end{tikzcd}$$

one deduces the commutative diagram

$$\begin{tikzcd}
T_1 \times G_1 \arrow[r] \arrow[d, "\varphi\times\rho"'] & T_1 \arrow[d, "\varphi"] \\
T_2 \times G_2 \arrow[r] & T_2
\end{tikzcd}$$

which first says that ρ is a group scheme morphism (since the actions are free) and finally that φ commutes with the actions of G_1 and G_2. $\square$

We are now ready to prove the following

Theorem 2.3. *The G-torsor $Y \to X_\eta$ can be extended to X if and only if the G_1-torsor $Y_1 \to X_\eta$ can be extended to X and the G_2-torsor $Y \to Y_1$ can be extended to Z_1 where $Z_1 \to X$ is a G_1'-torsor extending $Y_1 \to X_\eta$, for some S-model G_1' of G_1.*

Proof. The "only if" part is easy and left to the reader. So let us assume we are able to extend each component, i.e. there exist finite and flat S-group schemes G_1' and G_2' models (resp.) of G_1 and G_2, a G_1'-torsor $Z_1 \to X$ extending $Y_1 \to X_\eta$ and a G_2'-torsor $Z_2 \to Z_1$ extending $Y_2 \to Y_1$ (pointed resp. in $z_1 \in Z_1(S)$ and $z_2 \in Z_2(S)$, sections extending y_1 and y). Then we are in the situation described by the following diagram:

$$\begin{tikzcd}
Y = Y_2 \arrow[r] \arrow[d, "G_2"'] & Z_2 \arrow[d, "G_2'"] \\
Y_1 \arrow[r] \arrow[d, "G_1"'] & Z_1 \arrow[d, "G_1'"] \\
X_\eta \arrow[r] \arrow[d] & X \arrow[d] \\
Spec(K) \arrow[r] & S
\end{tikzcd} \tag{2}$$

In general $Z_2 \to X$ need not be a torsor, but from the tower $Z_2 \to Z_1 \to X$ we can obtain a torsor whose generic fibre is isomorphic to the G-torsor $Y \to X_\eta$. Indeed by a result of Garuti ([9], §2, Theorem 1) there exist flat S-group schemes of finite type N, M and H, a S-scheme T (provided with $t \in T(S)$) and morphisms $T \to Z_2$, $T \to Z_1$ and $T \to X$ which are respectively a N-torsor, M-torsor and H-torsor (all pointed), such that the following diagram commutes:

$$
\begin{tikzcd}
 & T \arrow[dl, "N"'] \arrow[ddl, "M"] \arrow[dddl, "H"] \\
Z_2 \arrow[d, "G'_2"'] & \\
Z_1 \arrow[d, "G'_1"'] & \\
X &
\end{tikzcd}
$$

then in particular there are canonical faithfully flat group scheme morphisms $\gamma_2 : M \to G'_2$ and $\gamma_1 : H \to G'_1$ over S where $M \simeq ker(\gamma_1)$ and $N \simeq ker(\gamma_2)$. First we observe that N is normal in H: indeed generically $N_\eta \trianglelefteq H_\eta$ because N_η is the kernel of the natural morphism $H_\eta \to G$; but N coincides with the schematic closure of N_η in H then $N \trianglelefteq H$ ([1], 1.2.5, Remarques d)). Hence we can construct the quotient H/N, which is a S-flat group scheme ([1] Théorème 4.C) that fits in the following exact sequence ([8], III, §3, n° 3, 3.7 a))

$$0 \longrightarrow G'_2 \longrightarrow H/N \longrightarrow G'_1 \longrightarrow 0 \tag{3}$$

then in particular H/N is finite since G'_2 and G'_1 are ([6], Proposition 9.2, (viii)). Let $\gamma : H \to (H/N)$ be the canonical faithfully flat morphism and let us consider the contracted product $Z := T \times^H (H/N)$ via γ which is a H/N-torsor. The contracted product commuting with base change ([8], III, §4, n° 3, 3.1), we have $Z_\eta := (T \times^H (H/N))_\eta \simeq T_\eta \times^{H_\eta} (H/N)_\eta \simeq T_\eta \times^{H_\eta} H_\eta/N_\eta$ (where the last isomorphism follows by [6], Proposition 9.2 (v)) then in particular $T_\eta \times^{H_\eta} H_\eta/N_\eta \simeq Y$ as a G-torsor over X (cf. Lemma 2.2) and Z is a pointed H/N-torsor over X extending the starting one. $\square$

Remark 2.4. Keeping notations of Theorem 2.3 we observe that Z factors through Z_1 and in particular $Z \to Z_1$ is a G'_2-torsor. Indeed

$$Z \times^{H/N} G'_1 \simeq (T \times^H H/N) \times^{H/N} G'_1 \simeq T \times^H G'_1 \simeq Z_1$$

then $Z \to Z_1$ is a $ker(H/N \to G'_1)$-torsor.

Corollary 2.5. *Let G be a finite and solvable K-group scheme and $Y \to X_\eta$ a pointed G-torsor. Then $Y \to X_\eta$ can be extended to a finite solvable G'-torsor $Z \to X$ for some model G' of G if and only if its commutative components can be extended.*

Proof. The case where G has a normal series of length $n = 2$ is exactly Theorem 2.3. With a little effort this procedure can be generalized to the case where G has no normal series of length $n = 2$, simply repeating Garuti's construction and Theorem 2.3. $\square$

3 Extension of solvable torsors

Let S and X be as in section 2.2. We recall a definition that we will often use from now on:

Definition 3.1. Assume X is endowed with a section $x \in X(S)$. A G-torsor $Y \to X$ over S, pointed at $y \in Y_x(S)$, is said to be quotient if X has a fundamental group scheme $\pi_1(X,x)$ (it always happens when X is connected according to [4]) and the canonical morphism of S-group schemes $\pi_1(X,x) \to G$ is faithfully flat.

Remark 3.2. Let S be the spectrum of a field. Then a pointed quotient G-torsor $Y \to X$ is always connected and if G is étale (it will always be the case in section 3.2) then the converse is also true.

In the situation of diagram (2) we now assume that G_1 and G_2 are commutative. In [3], Theorem 3.10 we have explained how to extend finite pointed quotient commutative torsors from X_η to X where X needs to satisfy some strong assumptions ([3], Notation 2.20). Thus for such X, it is not difficult to find a finite, flat and commutative S-group scheme G_1' as well as a G_1'-torsor $Z_1 \to X$ that extends the G_1-torsor $Y_1 \to X_\eta$. Unfortunately, even if we can easily find schemes X satisfying these strong conditions (loc. cit. §3.2), it is improbable that $Z_1 \to S$ satisfies the same assumptions. Consider for instance the case of curves: Theorem 3.10 of [3] holds when $X \to S$ is smooth, projective with geometrically integral fibres. However in general Z_1 needs not be smooth. So it is necessary to weaken the assumptions of [3] Theorem 3.10 and this will be done in Theorem 3.12. The idea is to replace in the statement *smooth* by *regular* so that if Z_1 is not regular we know at least how to make it regular (through the desingularization process) even if not smooth. This will be explained in section 3.1.

3.1 Commutative torsors over curves

Notation 3.3. Even if some results that we will recall hold in a more general setting from now till the end of the paper, if not stated otherwise, R will be a complete discrete valuation ring with fraction field K and residue field k. Moreover S will denote $Spec(R)$ while $\eta := Spec(K)$ and $s := Spec(k)$ will denote respectively the generic and special points (see section 1.2).

For the sake of completeness we recall in a few lines the definition of Néron model and some properties which will be used in this paper. The reader can refer to [7] for a deep discussion on the subject. Here we only consider Néron models of abelian varieties since it is the only case we will use.

Definition 3.4. Let A be an abelian variety over K. A Néron model of A is a smooth and separated R-scheme of finite type $\mathcal{N}_A$ whose generic fibre is isomorphic to A and which satisfies the following universal property (called the Néron mapping property): for each smooth R-scheme Y and each K-morphism

$u : Y_\eta \to A$ there exists a unique morphism $u' : Y \to \mathcal{N}_A$ extending u where as usual Y_η denotes the generic fibre of Y.

Proposition 3.5. *With notation as in Definition 3.4, then A admits a Néron model $\mathcal{N}_A$ over R.*

Proof. See for instance [7], §1.3, Corollary 2. □

By the Néron mapping property the Néron model $\mathcal{N}_A$ of A is unique up to canonical isomorphism and it is a commutative group scheme. Unfortunately in general $\mathcal{N}_A$ is not an abelian scheme and not even a semi-abelian scheme.

When $\mathcal{N}_A$ is an abelian scheme then we simply say that A has abelian (or good) reduction. If $\mathcal{N}_A$ is not an abelian scheme but there exists a finite Galois extension L/K such that the Néron model $\mathcal{N}_{A_L}$ of $A_L := A \times_{Spec(K)} Spec(L)$ is an abelian scheme over the integral closure R' of R in L then we say that A has potentially abelian (or potentially good) reduction.

Let X be an S-scheme and $X \to S$ a proper morphism of finite type, then in what follows we denote by $Pic_{(X/S)(fppf)}$ the sheaf, in the fppf topology, associated to the relative Picard functor given by

$$Pic_{X/S}(T) := Pic(X \times_S T)/Pic(T)$$

for any S-scheme T (see [3], §2 for a brief introduction and [13] for a complete reference on this topic)[1]. It is known that for any $s \in S$ the sheaf $Pic_{(X_s/k(s))(fppf)}$ is represented by a group scheme $\mathbf{Pic}_{X_s/k(s)}$ whose identity component is denoted by $\mathbf{Pic}^0_{X_s/k(s)}$; over S we denote by $Pic^0_{X/S}$ the subfunctor of $Pic_{(X/S)(fppf)}$ which consists to all elements whose restrictions to all fibres $X_s, s \in S$ belong to $\mathbf{Pic}^0_{X_s/k(s)}$. Now we recall both the well known definition of fibered surface and a result concerning the representability of $Pic^0_{X/S}$, for X a fibered surface.

Definition 3.6. Let $f : X \to S$ be a projective and flat morphism. We say that $f : X \to S$ is a fibered surface if X is an integral scheme of dimension 2. We say that $f : X \to S$ is a regular fibered surface if moreover X is regular.

Theorem 3.7. *Let $f : X \to S$ be a regular fibered surface with smooth generic fibre X_η and provided with a section $x \in X(S)$. Then $Pic^0_{X/S}$ is represented by a separated and smooth S-scheme $\mathbf{Pic}^0_{X/S}$ and coincides with the identity component of the Néron model of $J := \mathbf{Pic}^0_{X_\eta/K}$.*

Proof. First observe that X_η is geometrically integral (this will be recalled in Remark 3.15). Then observe that under these assumptions J is an abelian variety. According to [7], §9.5 Remark 5 the existence of a section implies that the greatest common divisor of the geometric multiplicities of the irreducible

[1] N.B.: here we have used Kleiman's notation. In [7], §8.1, Definition 2, however, our $Pic_{(X/S)(fppf)}$ is called "the relative Picard functor" and denoted $Pic_{X/S}$.

components of the special fibre X_s of X in X_s is one. Then by loc. cit. §9.5, Theorem 4 (b), $Pic^0_{X/S}$ is represented by a separated and smooth S-scheme $\mathbf{Pic}^0_{X/S}$ which coincides with the identity component of the Néron model of $J := \mathbf{Pic}^0_{X_\eta/K}$. $\square$

Let us denote by $u : X_\eta \to J$ the natural closed immersion usually known as the Abel-Jacobi map sending x_η to 0_J, where $J := \mathbf{Pic}^0_{X_\eta/K}$ is the Jacobian of X_η. In Proposition 3.10 we construct, when possible, a morphism $u' : X \to \mathcal{N}_J$ whose generic fibre is isomorphic to u, where $\mathcal{N}_J$ denotes the Néron model $\mathcal{N}_J$ of J.

In the following theorem, which is due to Lipman (cf. [14] and [5]), we recall the desingularization process for surfaces in the particular case of fibered surfaces with smooth generic fibre (see also [15], §8.3.4) that we will use several times:

Theorem 3.8. *Let Y be any fibered surface over S with smooth generic fibre Y_η. Let*

$$.. \to Y_i \to Y_{i-1} \to Y_{i-2} \to .. \to Y_1 \to Y_0 = Y \tag{4}$$

denote the sequence of blowing ups where for each i, the morphism $Y_i \to Y_{i-1}$ denotes

- *the normalization morphism if i is odd (it can be an isomorphism if Y_{i-1} is already normal);*
- *the blowing up at the singular points of Y_{i-1} if i is even.*

At each step, when i is even, the singular locus $Sing(Y_{i-1})$ of Y_{i-1} is a finite set of closed points contained in the special fibre $(Y_{i-1})_s$ while when i is odd then $Y_i \to Y_{i-1}$ is a finite morphism. Moreover there exists an integer $n \geq 0$ such that $\widetilde{Y} := Y_n$ is regular and the morphism $\widetilde{Y} \to Y$ is proper and generically an isomorphism.

Definition 3.9. We call the sequence $\widetilde{Y} = Y_n \to .. \to Y_1 \to Y_0 = Y$ the canonical desingularization of Y.

Proposition 3.10. *Let $f : X \to S$ be a regular fibered surface with smooth generic fibre X_η and provided with a section $x \in X(S)$. Let J be the Jacobian of X_η, $\mathcal{N}_J$ its Néron model and $u : X_\eta \to J$ the canonical closed immersion. Assume moreover that J has abelian reduction. Then there exists a morphism $u' : X \to \mathcal{N}_J$ whose generic fibre is isomorphic to u.*

Proof. If X were smooth this would be the Néron mapping property of the Néron model $\mathcal{N}_J$. Since in general this does not happen then we argue as follows: again we observe that X_η is geometrically integral; then by assumption $\mathcal{N}_J$ is an abelian scheme (thus proper), then construct the schematic closure $C := \overline{X_\eta}$ of X_η in $\mathcal{N}_J$, i.e. the only closed subscheme of $\mathcal{N}_J$, flat over S with generic fibre isomorphic to X_η. It is an integral scheme ([10] Proposition 9.5.9), proper over S (because $\mathcal{N}_J$ is) whose special fibre is equidimensional of dimension one ([15], Ch.

4, Proposition 4.16). Now we desingularize C, i.e. we construct a proper regular model (thus projective according to [15], Ch. 8, Theorem 3.16) $\widetilde{C}$ of X_η, flat over S ([15], Ch 4, Corollary 3.10) and a morphism $\widetilde{C} \to C$ which is generically an isomorphism. In particular $\mathcal{N}_J \simeq \mathcal{N}_J^0 \simeq \mathbf{Pic}^0_{X/S} \simeq \mathbf{Pic}^0_{\widetilde{C}/S}$ by Theorem 3.7 and from $\widetilde{C} \to \mathbf{Pic}^0_{X/S}$ one obtains the desired morphism $u' : X \to \mathcal{N}_J$. Indeed the morphism $\widetilde{C} \to \mathbf{Pic}^0_{X/S}$ is an element of $\mathbf{Pic}^0_{X/S}(\widetilde{C})$, then in particular this corresponds to an element

$$\xi \in \frac{Pic(X \times \widetilde{C})}{Pic(\widetilde{C})} = Pic_{X/S}(\widetilde{C}) = Pic_{(X/S)(fppf)}(\widetilde{C})$$

(use [15], Ch. 8, Corollary 3.6, (c) then apply [7], §8.1 Proposition 4) but since $\widetilde{C}$ and X are both regular then $Pic(X) \simeq Pic(X_\eta) \simeq Pic(\widetilde{C})$ and consequently

$$\frac{Pic(X \times \widetilde{C})}{Pic(\widetilde{C})} \simeq \frac{Pic(X \times \widetilde{C})}{Pic(X)} = Pic_{\widetilde{C}/S}(X) = Pic_{(\widetilde{C}/S)(fppf)}(X).$$

Hence, starting from ξ, we get a morphism $X \to Pic_{(\widetilde{C}/S)(fppf)}$ that on the generic and special fibres factors (resp.) through $u : X_\eta \to J$ and $X_s \to \mathbf{Pic}^0_{\widetilde{C}_s/k(s)}$ since X has geometrically connected fibres, thus obtaining a morphism $X \to \mathbf{Pic}^0_{\widetilde{C}/S}$ that composed with $\mathbf{Pic}^0_{\widetilde{C}/S} \simeq \mathcal{N}_J^0 \simeq \mathcal{N}_J$, gives the desired morphism $u' : X \to \mathcal{N}_J$ extending $u : X_\eta \to J$ as described by the following diagram:

$$\begin{tikzcd}
X_\eta \arrow[r] \arrow[dr, "u"] \arrow[dd] & X \arrow[d, dash] \arrow[dr, "u'"] & \\
 & J \arrow[r] \arrow[dl] \arrow[d] & \mathcal{N}_J \arrow[dl] \\
Spec(K) \arrow[r] & S &
\end{tikzcd}$$

□

A result due to Raynaud, that we state in our setting in the following theorem, shows that the hypothesis of Proposition 3.10 are satisfied in many relevant cases possibly after extending scalars:

Theorem 3.11. *Assume that $char(k) = p > 0$. Let X be a smooth fibered surface over R provided with a section $x \in X(S)$. Let G be a finite and étale K-group scheme of order p^n and $Y \to X_\eta$ a connected G-torsor over the generic fibre of X, then the Jacobian J_Y of Y has potential abelian reduction. In particular every commutative component Y_i of Y has a Jacobian J_{Y_i} with potential abelian reduction.*

Proof. It is known that G becomes constant after a finite Galois extension L of K. Moreover since X_η is integral, then it is connected and geometrically connected (this will be recalled in Remark 3.15); this implies that X_s is geometrically connected ([15], Ch. 8, Corollary 3.6, (b)), then finally use [19], Théorème 1. $\square$

Next theorem, that will be used later, concludes this section:

Theorem 3.12. *Let $f : X \to S$ be a regular fibered surface provided with a section $x \in X(S)$. Assume that f has smooth generic fibre $X_\eta \to \eta$. Assume moreover that the Jacobian J of X_η has abelian reduction. Then every finite, quotient, commutative torsor over X_η, pointed over x_η, can be extended to a finite commutative torsor over X, pointed over x.*

Proof. First we observe that X_η is geometrically connected. Now let $\mathcal{N}_J$ be the Néron model of J and $u' : X \to \mathcal{N}_J$ the Abel-Jacobi map constructed in Proposition 3.10. Let G be a finite and commutative K-group scheme, then according to [3], Corollary 3.8 we know that every finite, pointed (over $x_\eta \in X_\eta(K)$) quotient and commutative G-torsor $T' \to X_\eta$ is the pull back of a finite, pointed (over 0_J) quotient and commutative G-torsor $T \to J$. Now it is easy to find an R-model H of G (commutative, finite and flat) and a pointed (over $0_{\mathcal{N}_J}$), quotient H-torsor $Y \to \mathcal{N}_J$ whose generic fibre is isomorphic to $T \to J$ (cf. [2], §2.2). Then finally $Y' := Y \times_{\mathcal{N}_J} X$, the pull back over u', is a finite, commutative H-torsor over X (pointed over x) extending $T' \to X_\eta$. $\square$

Remark 3.13. In the situation of Theorem 3.12, if one is interested in extending only one fixed torsor $f : Y \to X_\eta$ pointed at $y \in Y(K)$, then he can take $f(y) \in X_\eta(K)$ and then its closure $\overline{f(y)} \in X(S)$. So it is not necessary to fix $x \in X(S)$ a priori since one can obtain a section $S \to X$ for any single torsor. This will be our point of view from now on without explicitly mentioning anymore.

3.2 Solvable torsors over curves

Notation 3.14. In this section R, S, K, k will be as in Notation 3.3. We ask moreover k to be algebraically closed and of positive characteristic $p > 0$: this will be used in Lemma 3.23 and not before. Furthermore from now on $f : X \to S$ will be a regular fibered surface with smooth generic fibre $X_\eta \to Spec(K)$. For any surjective S-scheme T the Néron blowing up of T at a closed subscheme C of T_s will be denoted by T^C: we refer the reader to [7], §3.2, [24], §1 or [1], §2.1 for the definition and properties.

In the following remark we recall some general properties that will be tacitly used in the reminder of this paper:

Remark 3.15. Let T be any S-scheme of finite type. If T_η is geometrically reduced (resp. geometrically irreducible) then T_η is reduced (resp. irreducible); if moreover $T(S) \neq \emptyset$ then its generic fibre T_η is geometrically connected if and only if it is connected ([15], Ch. 3, §2, ex. 2.11 and 2.13). Of course the same is true for the special fibre T_s. Finally we recall that T integral implies T_η integral and that if T is flat over S and T_η is integral then so is T ([10] Proposition 9.5.9).

Before stating the principal result we need some preliminary lemmas. Lemmas 3.16 and 3.17, as recalled in their proofs, slightly generalize [24] Lemma 1.3 and Theorem[2] 1.4, that we strongly use.

Lemma 3.16. *Let Y and $\widetilde{Y}$ be two schemes faithfully flat and of finite type over S and $h : \widetilde{Y} \to Y$ an affine model map. If the special fibre $h_s : \widetilde{Y}_s \to Y_s$ of h is a schematically dominant morphism (i.e. $\mathcal{O}_{Y_s} \hookrightarrow h_{s*}(\mathcal{O}_{\widetilde{Y}_s})$ is injective) then h is an isomorphism.*

Proof. Let $U = Spec(A)$ be any open affine subset of Y and $V = Spec(A') := h^{-1}(U)$ then consider $h_{|V} : V \to U$ and its special fibre $(h_{|V})_s : V_s \to U_s$ where $V_s = Spec(A'_k) = Spec(A' \otimes_R k)$, $U_s = Spec(A_k) = Spec(A \otimes_R k)$ and by assumption $A_k \hookrightarrow A'_k$. We are thus reduced to consider the affine case, then one just needs to argue as in [24], Lemma 1.3. □

Lemma 3.17. *Let Y and $\widetilde{Y}$ be two schemes faithfully flat and of finite type over S and $h : \widetilde{Y} \to Y$ an affine model map. Then h is isomorphic to a composite of a finite number of Néron blowing ups.*

Proof. If $h_s : \widetilde{Y}_s \to Y_s$ is schematically dominant the result follows from Lemma 3.16, otherwise consider the scheme theoretic image $C_1 := h_s(\widetilde{Y}_s)$ of $\widetilde{Y}_s$ in Y_s. It is a closed subscheme of Y_s ([10], §9.5). Now consider the Néron blowing up Y^{C_1} of Y in C_1 then h factors through $Y_1 := Y^{C_1}$. Denote by $h_1 : Y \to Y_1$ the S-morphism obtained. If its special fibre $(h_1)_s$ is schematically dominant then we are done otherwise we set $C_2 := (h_1)_s(\widetilde{Y}_s)$, $Y_2 := Y_1^{C_2}$ and so on. If Y and $\widetilde{Y}$ are affine then by means of [24] Theorem 1.4 we conclude that there exists $n \geq 0$ such that $\widetilde{Y} \simeq Y_n$. Otherwise $\widetilde{Y} \simeq \varprojlim_i Y_i$ and let $\{U_j\}_{j\in J}$ be an affine open cover of Y and $\left\{V_j := h^{-1}(U_j)\right\}_{j\in J}$ the induced affine open cover of $\widetilde{Y}$. Since Y is quasi compact we can take $|J| < \infty$. To give h is equivalent to give the family of morphisms

$$\{h_j := (V_j \xrightarrow{h_{|V_j}} U_j \longrightarrow Y)\}_{j\in J}$$

where we have given the V_j and U_j the induced subscheme structure ([12], II, Theorem 3.3, Step 3). For any $j \in J$ let us set $C_{1,j} := C_1 \times_Y U_j$ (i.e. the scheme theoretic image $(h_{|V_j})_s((V_j)_s)$, [10], Proposition 9.5.8) and $U_{1,j} := {U_j}^{C_{1,j}}$; the latter is isomorphic to $Y^{C_1} \times_Y U_j$ by means of the universal property of Néron blowing ups. Then we define $C_{2,j}$, $U_{2,j}$ and so on: it follows that $V_j \simeq \varprojlim_i U_{i,j}$ but since U_j and V_j are affine then the projective limits become stable after

[2] This result is stated by Waterhouse and Weisfeiler only for affine group schemes but, as observed by themselves, the group structure is never used ([24], page 552, Remark (4)).

$n(j) \geq 0$ steps. Take $n := max_{j \in J}\{n(j)\}$: this is the number of steps after which we can stop. $\square$

Lemma 3.18. *Let Y be a scheme faithfully flat and of finite type over S, C_2 a closed subscheme of Y_s and C_1 a closed subscheme of C_2. Denote by Y^{C_i} the Néron blowing up of Y in C_i $(i = 1, 2)$. Let $C' := (Y^{C_2})_s \times_{C_2} C_1$ the induced closed subscheme of $\left(Y^{C_2}\right)_s$ then $Y^{C_1} \simeq \left(Y^{C_2}\right)^{C'}$.*

Proof. This follows directly from the universal property of the Néron blowing up and the following diagram:

$$\begin{tikzcd}
(Y^{C_1})_s \arrow[rr, dashed] \arrow[dr] & & C' \arrow[rr, hook] \arrow[dl] & & (Y^{C_2})_s \arrow[dl] \\
& C_1 \arrow[rr, hook] \arrow[dr, hook] & & C_2 \arrow[dl, hook] & \\
& & Y_s & &
\end{tikzcd}$$

$\square$

Lemma 3.19. *Let Y be a fibered surface. Let $f : Y \to X$ be a finite and flat morphism, C a closed subscheme of Y_s and Y^C the Néron blowing up of Y in C. Then there exist a regular fibered surface X' and an affine model map $X' \to X$ such that $Y^C \simeq Y \times_X X'$.*

Proof. Let $f_s : Y_s \to X_s$ be the special fibre of f and $D_1 := f_s(C)$ the scheme theoretic image of C: it is a closed subscheme of X_s. Now consider the fibre product $C_1 := D_1 \times_{X_s} Y_s$ and the natural closed immersion $C \hookrightarrow C_1$: if it is an isomorphism then, by the universal property of the Néron blowing up, $Y^C \simeq X^{D_1} \times_X Y$ hence $X' := X^{D_1}$ is the required solution. Otherwise let $Y_1 := Y^{C_1}$, $X_1 := X^{D_1}$ and $f_1 : Y_1 \to X_1$ the pull back of f over $X_1 \to X$. The morphism $Y^C \to Y$ now factors through Y_1; then we analyze the morphism $Y^C \to Y_1$: by Lemma 3.18 $Y^C \simeq Y_1^{C_1'}$ where $C_1' := C \times_{C_1} \left(Y_1\right)_s$, thus we are in the same situation as before: let $D_2 := \left(f_1\right)_s(C_1')$, $C_2 := D_2 \times_{\left(X_1\right)_s} \left(Y_1\right)_s$, $Y_2 := Y_1^{C_2}$, $X_2 := X_1^{D_2}$, $C_2' := C_1' \times_{C_2} \left(Y_2\right)_s$ and so on. We finally obtain the isomorphism $Y^C \simeq \varprojlim_i Y_i$ (where we have set $Y_0 := Y$). Now using arguments similar to those used in the last part of the proof of Lemma 3.17 we are reduced to study the case where X (then also Y and Y^C) is affine: so let us set $Y_i := Spec(A_i)$ and $Y^C = Spec(B)$ then since every A_i is integral ([10], Corollaire 1.2.7) the morphisms $Y_i \to Y_{i-1}$ induce a sequence of inclusions

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq .. \subseteq A_i \subseteq .. \subseteq B;$$

and equality occurs at some finite stage because B is finitely generated. Hence $Y^C \simeq Y_n$ and $X' := X_n$ allows us to conclude. $\square$

Remark 3.20. In Lemma 3.19 we never use the assumption that the absolute dimension of X and Y is 2, but it is the only case of interest in this paper.

Lemma 3.21. *Let $f : Y \to X$ be a finite and flat morphism with Y integral. Let $h : \widetilde{Y} \to Y$ be an affine model map. Then there exist a regular fibered surface $\widetilde{X}$ and an affine model map $\widetilde{X} \to X$ such that $\widetilde{Y} \simeq Y \times_X \widetilde{X}$. Moreover $\widetilde{X} \to X$ is isomorphic to a composite of a finite number of Néron blowing ups.*

Proof. This is just a consequence of Lemmas 3.17 and 3.19. □

Corollary 3.22. *Let $f : Y \to X$ be a finite and flat morphism with Y integral. Assume that the generic fibre Y_η of Y is smooth and geometrically integral. Let $h : \overline{Y} \to Y$ be the normalization morphism. Then there exist a regular fibered surface $\overline{X}$ and a finite model map $\overline{X} \to X$ such that $\overline{Y} \simeq Y \times_X \overline{X}$.*

Proof. In this context the normalization morphism $h : \overline{Y} \to Y$ is a finite (then affine) model map (cf. Theorem 3.8). Then the result is just a consequence of Lemma 3.21. □

Lemma 3.23. *Let G be a finite and flat S-group scheme with infinitesimal special fibre G_s and $f : Y \to X$ a G-torsor. Assume that the generic fibre Y_η of Y is smooth and geometrically integral. Let moreover $\widetilde{Y} \to Y$ be the blowing-up of Y centered at a closed point q of the special fibre Y_s of Y. Then $\widetilde{Y} \simeq Y \times_X \widetilde{X}$ where $\widetilde{X} \to X$ is the blowing up centered at $p := f_s(q)$.*

Proof. The residue field k being algebraically closed then $q : Spec(k) \to Y$ and also $p : Spec(k) \to X$ are k-rational points ([15], Ch.2 ex. 5.9). Thus, since $f_s : Y_s \to X_s$ is a G_s-torsor, $Y_p := Spec(k) \times_X Y \simeq G_s$ and the canonical closed immersion $q \to G_s$ identifies q with $(G_s)_{\mathrm{red}}$ (recall that G_s is infinitesimal). Then the blowing up Y' of Y centered at Y_p is isomorphic to the blowing up of Y centered at q ([15], Ch. 2, ex. 3.11 (a)). But since $Y' \simeq Y \times_X \widetilde{X}$ ([15], Ch. 8, Proposition 1.12 (c)) then $\widetilde{Y} \simeq Y \times_X \widetilde{X}$, as required. □

Proposition 3.24. *Let G be a finite and flat S-group scheme with infinitesimal special fibre G_s and $f : Y \to X$ a G-torsor. Assume that the generic fibre Y_η of Y is smooth and geometrically integral. Let moreover $\widetilde{Y} \to Y$ be the canonical desingularization of Y. Then there exist a regular fibered surface $\widetilde{X}$ and a morphism $\widetilde{X} \to X$ such that $\widetilde{Y} \simeq Y \times_X \widetilde{X}$. In particular $\widetilde{Y} \to \widetilde{X}$ is a G-torsor.*

Proof. According to Theorem 3.8 $\widetilde{Y} \to Y$ is a sequence of normalization morphisms (which are finite morphisms) and blowing ups centered at a finite set of singular closed points. Then in order to conclude it is sufficient to use Lemma 3.23 and Corollary 3.22. □

Before stating the main theorem of this paper we need a last lemma:

Lemma 3.25. *Let $Z \to X$ be a finite $(\mathbb{Z}/p\mathbb{Z})_R$-torsor. Then there exist a finite and flat R-group scheme G with infinitesimal special fiber, a G-torsor $Y \to X$ and a model map $\varphi : Z \to Y$ commuting with the actions of $(\mathbb{Z}/p\mathbb{Z})_R$ and G.*

Proof. That a model map $\rho : (\mathbb{Z}/p\mathbb{Z})_R \to G$ such that G_s is infinitesimal exists is clear from [16], §3.2 when $char(K) = p$ and from [18], I, §2, when $char(K) = 0$, then the model map $\varphi : Z \to Y$ is given by the contracted product (through ρ) $Y = Z \times^{(\mathbb{Z}/p\mathbb{Z})_R} G$. □

Remark 3.26. The G-torsor $Y \to X$ obtained in Lemma 3.25 has trivial special fibre but this will not affect the following discussion.

Theorem 3.27. *Let X be a smooth fibered surface over R. Let G be a finite, étale, solvable K-group scheme of order p^n and $Y \to X_\eta$ a connected G-torsor, pointed at $y \in Y(K)$. Then, possibly after a finite extension of scalars, there exist a regular fibered surface $\widetilde{X}$, a model map $\widetilde{X} \to X$, a finite flat and solvable R-group scheme G' such that $Y \to X_\eta$ can be extended to a G'-torsor $Y' \to \widetilde{X}$. Moreover we can construct Y' in such a way to make it regular.*

Proof. First of all we observe that we can decompose $Y \to X_\eta$ into a tower of n torsors $Y_1 \to X_\eta$, $Y_i \to Y_{i-1}$ (for $i = 2, .., n$, where $Y_n = Y$) each one being a quotient pointed G_i-torsor where $|G_i| = p$. Possibly after extending scalars we can assume that $G_i \simeq (\mathbb{Z}/p\mathbb{Z})_K$ (for all $i = 1, .., n$) and that the Jacobian J_{Y_i} has abelian reduction (Theorem 3.11). Assume first that $n = 2$: according to Theorem 3.12 there exist a finite and flat R-group scheme G'_1 of order p, generically isomorphic to G_1, and a G'_1-torsor $Z_1 \to X$ extending $Y_1 \to X_\eta$. We can assume by Lemma 3.25 that $(G_1)_s$ is infinitesimal. If Z_1 is regular we go on extending $Y_2 \to Y_1$, otherwise we desingularize Z_1 as recalled in Theorem 3.8, i.e. we find a regular fibered surface Z'_1 and a model map $Z'_1 \to Z_1$. Moreover by Proposition 3.24 there exist a regular fibered surface X' and a model map $X' \to X$ such that $Z'_1 \to X'$ is a G'_1-torsor. Now we proceed as before: there exist a finite and flat R-group scheme G'_2 of order p, generically isomorphic to G_2, and a G'_2-torsor $Z_2 \to Z'_1$ extending $Y_2 \to Y_1$. Again we can assume that $(G_2)_s$ is infinitesimal. Then by Theorem 2.3 there exist a finite, flat S-group scheme G' generically isomorphic to G, with infinitesimal special fibre and a G'-torsor $Z \to X'$ extending $Y \to X_\eta$ and we are done setting $Y' := Z$. We only mention how to proceed when $n > 2$: we start from Z and, as before, we desingularize it, i.e. we find a regular fibered surface Z'_2 and a model map $Z'_2 \to Z$. As before there exist a regular fibered surface $X'' \to X'$ such that $Z'_2 \to X''$ is a G'_1-torsor; then we can extend $Y_3 \to Y_2$ to a torsor over Z'_2 and so on. We argue in the same way to prove that we can find a regular Y' (if it is not we desingularize, etc.). □

Corollary 3.28. *Let X be a smooth fibered surface over R. Let G be a finite, étale, K-group scheme having a normal series of length $n = 2$. Let $Y \to X_\eta$ be a connected G-torsor, pointed in $y \in Y(K)$. Then, possibly after a finite extension of scalars, there exist a regular fibered surface $\widetilde{X}$, a model map $\widetilde{X} \to X$, a finite* *flat and solvable R-group scheme G' such that $Y \to X_\eta$ can be extended to a G'-torsor over $\widetilde{X}$.*

Proof. We can assume that the K-group scheme G is constant (it is always true possibly after extending scalars). Let us decompose $Y \to X_\eta$ into a tower of two commutative torsors: a G_1-torsor $Y_1 \to X_\eta$ and a G_2-torsor $Y \to Y_1$. If $p \nmid |G_1|$ then the problem has an easy answer (see the introduction), otherwise let p^n be the maximal p-power dividing $|G_1|$ and ${}^pG_1$ a (normal) K-subgroup of G_1 of order p^n. Then the Jacobian J_{Y_1} of Y_1 has potentially abelian reduction. Indeed Y_1 can be decomposed into a tower of two torsors: a ${}^pG_1$-torsor $Y_1 \to T$ and a $G_1/{}^pG_1$-torsor $T \to X_\eta$. The latter can be extended, possibly after extending scalars, to a finite and étale torsor $T' \to X$ (we refer the reader to the introduction of this paper) then we apply Theorem 3.11 to $Y_1 \to T$. Now we forget this decomposition for $Y_1 \to X_\eta$ and we assume that over K the Jacobian J_{Y_1} has abelian reduction (we have seen it is always true possibly after extending scalars). We would rather consider the following decomposition for $Y_1 \to X_\eta$ as a tower of two torsors: a ${}^pG_1$-torsor $P \to X_\eta$ and a $G_1/{}^pG_1$-torsor $Y_1 \to P$. Theorem 3.11 tells us that the Jacobian J_P of P has potentially abelian reduction; again we can assume that it has in fact abelian reduction. Hence according to Theorem 3.27 there exist a regular fibered surface $\widetilde{X}$, a model map $\widetilde{X} \to X$, a finite flat and commutative R-group scheme H_1 such that $P \to X_\eta$ can be extended to a H_1-torsor $P' \to \widetilde{X}$ with P' regular. Furthermore by Theorem 3.12 there exists a finite flat and commutative R-group scheme H_2 such that $Y_1 \to P$ can be extended to a H_2-torsor $Y_1' \to P'$; by Theorem 2.3 there exist a finite and flat S-group scheme H generically isomorphic to G_1 and a H-torsor $Z \to \widetilde{X}$ extending $Y_1 \to X_\eta$. Since $p \nmid |H_2|$ then H_2 is étale; moreover $Z \to \widetilde{X}$ factors through P', more precisely $Z \to P'$ is a H_2-torsor (Remark 2.4) so $Z \to P'$ is smooth, then Z is regular as P' is (see for instance [7], §2.3 Proposition 9). Finally we can apply again Theorem 3.12 to $Y \to Y_1$ and 2.3 in order to conclude. $\square$

Remark 3.29. It is obvious that the tools we have presented allows us to extend solvable torsors even if they do not have a normal series of length 2 but only in some particular cases, for example if every commutative component Y_i of the torsor $Y \to X_\eta$ has a Jacobian J_{Y_i} that has potentially abelian reduction. As clear from the proof of Corollary 3.28 this condition is satisfied, for instance, when all the G_i but G_1 have order not divisible by p.

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Marco Antei
Dept. of Mathematical Sciences,
Korea Advanced Institute of Science and Technology
Yuseong-gu, Daejeon 305-701, Republic of Korea
E-mail: marco.antei@gmail.com