On the Grothendieck-Lefschetz Theorem for a Family of Varieties

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Abstract. Let k be an algebraically closed field of characteristic $p > 0$, W the ring of Witt vectors over k and R the integral closure of W in the algebraic closure $\overline{K}$ of $K := Frac(W)$; let moreover X be a smooth, connected and projective scheme over W and H a relatively very ample line bundle over X. We prove that when $dim(X/W) \geq 2$ there exists an integer d_0, depending only on X, such that for any $d \geq d_0$, any $Y \in |H^{\otimes d}|$ connected and smooth over W and any $y \in Y(W)$ the natural R-morphism of fundamental group schemes $\pi_1(Y_R, y_R) \to \pi_1(X_R, y_R)$ is faithfully flat, X_R, Y_R, y_R being respectively the pull back of X, Y, y over $Spec(R)$. If moreover $dim(X/W) \geq 3$ then there exists an integer d_1, depending only on X, such that for any $d \geq d_1$, any $Y \in |H^{\otimes d}|$ connected and smooth over W and any section $y \in Y(W)$ the morphism $\pi_1(Y_R, y_R) \to \pi_1(X_R, y_R)$ is an isomorphism.

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Contents

1 Introduction

The notion of fundamental group scheme of a connected and reduced scheme over a perfect field has been introduced by Madhav Nori in [13] and [14] as the affine group scheme over k naturally associated to the tannakian category of essentially finite vector bundles. Then in [3] it has been generalized by Gasbarri for integral schemes over a connected Dedekind scheme. In this latter description, however, the tannakian tool used by Nori is absent. This has been added by the second (alphabetically) author and Subramanian in [12] where they describe the fundamental group scheme of a smooth and projective scheme over a certain Prüfer ring R (more details will be recalled in section 3) by means of tannakian lattices introduced by Wedhorn in [15].

Now let Z be a smooth and projective variety over an algebraically closed field k. If Y is any smooth ample hypersurface on Z and y any point of Y then by Grothendieck-Lefschetz theory (cf. [4], Exposé X) we know that the induced group homomorphism between the étale fundamental groups

$$\pi_1^{\text{ét}}(Y, y) \to \pi_1^{\text{ét}}(Z, y)$$

is surjective when $dim(Z) \geq 2$ and an isomorphism when $dim(Z) \geq 3$, thus in particular if $char(k) = 0$ the same result automatically holds for the fundamental group scheme (even when k is not algebraically closed and this can be seen using the fundamental short exact sequence). When $char(k) = p > 0$ then in [10] and [1] it has been proved, independently, that theorems of Grothendieck-Lefschetz type hold in the following formulation: let H be a very ample line bundle over Z then when $dim(Z) \geq 2$ the natural homomorphism $\widehat{\varphi} : \pi_1(Y, y) \to \pi_1(Z, y)$ between fundamental group schemes induced by the inclusion map $\varphi : Y \hookrightarrow Z$ is faithfully flat whenever Y is in the complete linear system $|H^{\otimes d}|$ for any integer $d \geq d_0$ where d_0 is an integer depending only on Z. If moreover $dim(Z) \geq 3$, then $\widehat{\varphi}$ is an isomorphism whenever Y is in the complete linear system $|H^{\otimes d}|$ for any integer $d \geq d_1$ where d_1 is an integer depending only on Z.

Let finally k be an algebraically closed field of characteristic $p > 0$, W the ring of Witt vectors over k and R the integral closure of W in the algebraic closure $\overline{K}$ of $K := Frac(W)$, in this paper we prove the following generalization (where the subscript $_R$ will denote the pull back over $Spec(R)$):

Theorem 1.1. *(Cf. Theorems 3.2 and 3.3) Let X be a smooth and projective scheme over W and H a relatively very ample line bundle over X.*

1. *If $dim(X/W) \geq 2$ then there exists an integer d_0 (depending only on X) such that for any $d \geq d_0$, any $Y \in |H^{\otimes d}|$ connected and smooth over W and any $y \in Y(W)$ the natural R-morphism of fundamental group schemes $\pi_1(Y_R, y_R) \to \pi_1(X_R, y_R)$ is faithfully flat.*

2. *If moreover $dim(X/W) \geq 3$ then there exists an integer d_1 (depending only on X) such that for any $d \geq d_1$, any $Y \in |H^{\otimes d}|$ connected and smooth over W and any $y \in Y(W)$ the morphism $\pi_1(Y_R, y_R) \to \pi_1(X_R, y_R)$ is an isomorphism.*

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2 Vanishing lemmas

Notation 2.1. Throughout this section k will be an algebraically closed field of positive characteristic p and X a smooth and projective scheme over k.

In section 3 we will strongly make use of the vanishing Lemmas 2.3 and 2.8 whose idea is already contained in [10], lemmas 2.2 and 3.3. Let H be a very ample line bundle on X. Moreover let $F_X : X \to X$ be the absolute Frobenius morphism, then we denote by F_X^m the m-th iterate of F_X. We recall that a vector bundle V over X is essentially finite (cf. [13] and [14] for Nori's definition) if there exists a finite k-group scheme G and a principal G-bundle $\pi : E \to X$ such that $\pi^*(V)$ is trivial on E. We also recall (cf. [11]) that a vector bundle V over X is called F-trivial if there exists an integer $n \geq 0$ such that $F_X^{n*}(V)$ is trivial on X. When V is essentially finite then there exists an integer $n \geq 0$ such that $F_X^{n*}(V)$ is Galois étale trivial (cf. [11]), i.e. there exists a Galois étale covering $\pi' : E' \to X$ such that $\pi'^*(F_X^{n*}(V))$ is trivial on E' (the converse is also true). We denote by $EF(X)$ the tannakian category of essentially finite vector bundles over X. We finally recall that $V \in EF(X)$ if and only if the dual $V^* \in EF(X)$. We will often use the notation $V^{(p^m)} = F_X^{m*}(V)$ for every integer $m \geq 0$ for the comfort of the reader.

Remark 2.2. Let $\Omega_X^\bullet$ be the De Rham complex and let us set $B_X^j := Im(d : \Omega_X^{j-1} \to \Omega_X^j)$ and $Z_X^j := Ker(d : \Omega_X^j \to \Omega_X^{j+1})$ for all $j > 0$, then we can define the Cartier operator

$$C_X : Z_X^\bullet \to \Omega_X^\bullet$$

whose kernel is $B_X^\bullet$; in particular we have the following exact sequences of vector bundles

$$0 \to B_X^1 \to Z_X^1 \to \Omega_X^1 \to 0, \tag{1}$$

$$0 \to \mathcal{O}_X \to F_{X*}(\mathcal{O}_X) \to B_X^1 \to 0, \tag{2}$$

$$0 \to Z_X^1 \to F_{X*}(\Omega_X^1) \to B_X^2 \to 0. \tag{3}$$

Lemma 2.3. *Assume $dim(X) \geq 2$. Let V be any essentially finite vector bundle over X, then there exists a uniform (i.e. depending only on X) positive integer n_0 such that $H^1(X, V(-n))$ vanishes for all $n > n_0$.*

Proof. (See also [1], Lemma 4.7.) Let us consider the exact sequence (2):

$$0 \to \mathcal{O}_X \to F_{X*}(\mathcal{O}_X) \to B_X^1 \to 0$$

and tensor it by $V(-n)$; then we have the induced long exact sequence

$$.. \to H^0(X, V(-n) \otimes B_X^1) \to H^1(X, V(-n)) \to H^1(X, F_{X*}(\mathcal{O}_X) \otimes V(-n)) \to ..$$

By the projecton formula we have ${F_X}_*(\mathcal{O}_X) \otimes V(-n) \simeq {F_X}_*({F_X}^*(V(-n)))$ thus $H^1(X, {F_X}_*(\mathcal{O}_X) \otimes V(-n)) \simeq H^1(X, {F_X}^*(V(-n))) = H^1(X, V^p(-np))$. Moreover

$$H^0(X, V(-n) \otimes B_X^1) \simeq \mathcal{H}om(V^*(n), B_X^1)(X) \simeq Hom(V^*(n), B_X^1)$$

and $Hom(V^*(n), B_X^1) = 0$ as soon as $\mu(V^*(n)) > \mu_{max}(B_X^1)$. Set $n_0 := \mu_{max}(B_X^1)$, so that previous inequality becomes $n > n_0$. Thus, when $n > n_0$, the morphism

$$H^1(X, V(-n)) \to H^1(X, V^p(-np))$$

becomes injective. Iterating the process (tensor (2) by $V^p(-np)$ and so on) we obtain the injective morphism

$$H^1(X, V(-n)) \to H^1(X, V^{(p^m)}(-np^m)) \tag{4}$$

for every $m \geq 0$ and every $n > n_0$ where, we recall, $n_0 = \mu_{max}(B_X^1)$ depends only on X. We have already recalled that since $V \in EF(X)$ then there exists an integer $l \geq 0$ such that $F_X^{l*}(V)$ is Galois étale trivial, then so are its stable components $V_i, i = 1, .., L$, where L is the length of its Jordan-Hölder filtrations. Thus every V_i is stable and Galois étale trivial then by [2], Théorème 2.3.2.4 for every i there exists an integer $t_i > 0$ such that $F_X^{t_i *}(V_i) \simeq V_i$. This implies that the isomorphism classes of stable components of the vector bundles in the family $\{F_X^{t*}(V)\}_{t>0}$ are only finitely many. Denote these isomorphism classes of stable vector bundles by $\{W_j\}_{j \in J}$, then by the Enriques-Severi-Zariski-Serre vanishing lemma and the fact that $|J| < +\infty$ we know that $H^1(X, W_j(-np^s)) = 0$ for all $s >> 0$ then $H^1(X, V^{(p^m)}(-np^m))$ for all $m >> 0$. Now remember that for $n > n_0$ we have the injection (4) hence $H^1(X, V(-n)) = 0$. □

Remark 2.4. Using arguments similar to those that in Lemma 2.3 allowed us to prove that $H^1(X, V^{(p^m)}(-np^m))$ vanishes for all $m >> 0$ one also proves that $H^2(X, V^{(p^t)}(-np^t))$ vanishes for all $t >> 0$.

The aim of the reminder of this section is to prove a vanishing result for the group $H^2(X, V(-n))$. This will be done in Lemma 2.8, but we first need some preliminary steps. Let us start with a lemma which is also proved in [1], Proposition 4.11:

Lemma 2.5. *Assume $dim(X) \geq 2$. Let V be any essentially finite vector bundle over X, then there exists a uniform positive integer n_1 such that $H^1(X, \Omega_X^1 \otimes V(-n))$ vanishes for all $n > n_1$.*

Proof. Let $\mathcal{T}_X$ be the tangent bundle of X, then $\mathcal{T}_X(s)$ is generated by a finite number of global sections for some integer s depending only on X, then we have an exact sequence

$$0 \to S^* \to \mathcal{O}_X^N \to \mathcal{T}_X(s) \to 0 \tag{5}$$

for some vector bundle S; dualizing we obtain

$$0 \to \Omega^1_X(-s) \to \mathcal{O}_X^N \to S \to 0 \tag{6}$$

then we tensor by $V(-m)$ for some positive integer m and we get the following exact sequence

$$0 \to \Omega^1_X(-s) \otimes V(-m) \to \mathcal{O}_X^N \otimes V(-m) \to S \otimes V(-m) \to 0; \tag{7}$$

consider the induced long exact sequence

$$H^0(X, S \otimes V(-m)) \to H^1(X, \Omega^1_X \otimes V(-m-s)) \to H^1(X, V(-m))^N \tag{8}$$

Now $H^0(X, S \otimes V(-m)) = 0$ as soon as $m > \mu_{max}(S)$ and $s_0 := \mu_{max}(S)$ is clearly independent from V. Moreover by Lemma 2.3 there exists n_0 dependent only on X such that $H^1(X, V(-m))^N = 0$ for all $m > n_0$. Thus $H^1(X, \Omega^1_X \otimes V(-m-s)) = 0$ for $m > max\{n_0, s_0\}$. We have thus proved that there exists an integer n_1 such that $H^1(X, \Omega^1_X \otimes V(-n)) = 0$ for all $n > n_1$. □

Remark 2.6. Assume $dim(X) \geq 2$. Let V be any essentially finite vector bundle over X, n an integer such that $n > n_1$, as defined in Lemma 2.5 then $H^1(X, F_{X*}(\Omega^1_X) \otimes V(-n))$ vanishes. Indeed $F_{X*}(\Omega^1_X) \otimes V(-n) \simeq F_{X*}((V^p(-np)) \otimes \Omega^1_X)$ thus $H^1(X, F_{X*}(\Omega^1_X) \otimes V(-n)) \simeq H^1(X, (V^p(-np)) \otimes \Omega^1_X)$ and the latter is trivial by Lemma 2.5, V^p being essentially finite.

Lemma 2.7. *Assume $dim(X) \geq 2$. Let V be any essentially finite vector bundle over X, then there exists a uniform positive integer n_2 such that $H^1(X, V(-n) \otimes B^1_X)$ vanishes for all $n > n_2$.*

Proof. We tensor (cf. Remark 2.2)

$$0 \to B^1_X \to Z^1_X \to \Omega^1_X \to 0$$

by $V(-n)$ then we get the long exact sequence

$$.. \to H^0(X, \Omega^1_X \otimes V(-n)) \to H^1(X, B^1_X \otimes V(-n)) \to H^1(X, Z^1_X \otimes V(-n)) \to ..$$

but $H^0(X, \Omega^1_X \otimes V(-n)) = Hom(V^*(n), \Omega^1_X) = 0$ as soon as $n > \mu_{max}(\Omega^1_X)$ where $r_1 := \mu_{max}(\Omega^1_X)$ is independent of V. Thus

$$H^1(X, B^1_X \otimes V(-n)) \hookrightarrow H^1(X, Z^1_X \otimes V(-n)) \tag{9}$$

is injective for all $n > r_1$. In a similar way, tensoring

$$0 \to Z^1_X \to F_{X*}(\Omega^1_X) \to B^2_X \to 0$$

by $V(-n)$ we obtain the injection

$$H^1(X, Z^1_X \otimes V(-n)) \hookrightarrow H^1(X, F_{X*}(\Omega^1_X) \otimes V(-n)) \tag{10}$$

for all $n > r_2$, with $r_2 := \mu_{max}(B^2_X)$. Combining (9) and (10) we obtain the injection

$$H^1(X, B^1_X \otimes V(-n)) \hookrightarrow H^1(X, F_{X*}(\Omega^1_X) \otimes V(-n))$$

for all $n > max\{r_1, r_2\}$. But the group $H^1(X, F_{X*}(\Omega^1_X) \otimes V(-n))$ is trivial for all $n > n_1$, according to Remark 2.6. Now let us set $n_2 := max\{r_1, r_2, n_1\}$, then for all $n > n_2$ we have $H^1(X, B^1_X \otimes V(-n)) = 0$, as required.

□

Lemma 2.8. *Assume $dim(X) \geq 3$. Let V be any essentially finite vector bundle over X, then there exists a uniform positive integer n_2 such that $H^2(X, V(-n))$ vanishes for all $n > n_2$.*

Proof. (See also [1], Lemma 4.12.) By Remark 2.2 we have the exact sequence

$$0 \to \mathcal{O}_X \to F_{X*}(\mathcal{O}_X) \to B^1_X \to 0$$

that we tensor by $V(-n)$; then we have the induced long exact sequence

$$.. \to H^1(X, V(-n) \otimes B^1_X) \to H^2(X, V(-n)) \to H^2(X, V^p(-np)) \to ..$$

According to Lemma 2.7 there exists a uniform integer n_2 (independent of V) such that for all $n > n_2$ the group $H^1(X, V(-n) \otimes B^1_X)$ vanishes then we have the injection $H^2(X, V(-n)) \hookrightarrow H^2(X, V^p(-np))$ and, iterating,

$$H^2(X, V(-n)) \hookrightarrow H^2(X, V^{(p^t)}(-np^t));$$

but $H^2(X, V^{(p^t)}(-np^t)) = 0$ for $t >> 0$ (cf. Remark 2.4) then finally $H^2(X, V(-n)) = 0$ for all $n > n_2$ and we are done. □

3 Theorems

Notation 3.1. Throughout this section k will be, as before, an algebraically closed field of positive characteristic p. Furthermore $W := W(k)$ will denote the ring of Witt vectors over k and R the integral closure of W in the algebraic closure $\overline{K}$ of $K := Frac(W)$. Till the end of the paper X will denote a connected, smooth and projective scheme over W. Everywhere the subscript $_R$ will denote the pull back over $Spec(R)$.

For any finite extension K' of K we will denote by W' the integral closure of W in K' (with residue field k) and by X' the fibered product $X \times_W W'$. An essentially finite vector bundle over X' has been defined in [12] as a vector bundle V over X' whose restrictions V_k and $V_{K'}$ respectively to $(X')_k := X' \times_{W'} k \simeq X_k$ and $(X')_{K'} := X' \times_{W'} K'$ are essentially finite in the usual sense. Let us fix a W-valued point $x \in X(W)$ and let x_R be the induced R-valued point on X_R. Let $\mathcal{L}$ be the full subcategory of $Coh(X_R)$ (coherent sheaves over X_R) whose objects are defined as follows: $V \in Ob(Coh(X_R))$ belongs to $Ob(\mathcal{L})$ if and only if there exists a finite extension K' of K and an essentially finite vector bundle V' over X', such that V is pull back of V' over X_R. The category $\mathcal{L}$ provided with the fiber functor $x_R^* : \mathcal{L} \to R\text{-}mod$ is a tannakian lattice as defined by Wedhorn in [15]. We denote by $\pi_1(X_R, x_R)$ the affine R-group scheme associated to it which is the fundamental group scheme of X_R. Now let x' be the section on X' induced by x. A principal bundle E over X', pointed over x', is called Nori-reduced if $H^0(E, \mathcal{O}_E) = W'$.

Theorem 3.2. *Let X be a smooth, connected and projective scheme over W of relative dimension $dim(X/W) \geq 2$. Let H be a relatively very ample line bundle on X. Then there exists an integer d_0 (depending only on X) such that for any $d \geq d_0$, any $Y \in |H^{\otimes d}|$ connected and smooth over W and any section $y \in Y(W)$ the homomorphism $\widehat{\varphi} : \pi_1(Y_R, y_R) \to \pi_1(X_R, y_R)$ induced by the closed immersion $\varphi : Y \hookrightarrow X$ is faithfully flat.*

Proof. As before let K' be any finite extension of K, W' the integral closure of W in K' and X' (resp. Y') the fibered product $X \times_W W'$ (resp. $Y \times_W W'$). It is sufficient to prove that for any Nori-reduced principal bundle $e : E \to X'$ its restriction to Y', denoted $e_{Y'} : E_{Y'} \to Y'$, is still Nori-reduced. So we are assuming that $H^0(E, \mathcal{O}_E) = H^0(X', e_*(\mathcal{O}_E)) = W'$. We tensor by $e_*(\mathcal{O}_E)$ the exact sequence

$$0 \to \mathcal{O}_{X'}(-d) \to \mathcal{O}_{X'} \to \mathcal{O}_{Y'} \to 0$$

and we consider the long exact sequence associated

$$.. \to H^0(X', e_*(\mathcal{O}_E)) \to H^0(Y', e_{Y'*}(\mathcal{O}_{E_{Y'}})) \to H^1(X', e_*(\mathcal{O}_E)(-d)) \to ..$$

by Lemma 2.3 there exists a uniform d_0 (depending only on X_k) such that $H^1(X_k, e_{k*}(\mathcal{O}_{E_k})(-d)) = 0$ for all $d \geq d_0$ and where E_k denotes the special fiber of E. This implies that $H^1(X', e_*(\mathcal{O}_E)(-d))$ vanishes for all $d \geq d_0$. Thus $E_{Y'} \to Y'$ is Nori-reduced as $H^0(Y', e_{Y'*}(\mathcal{O}_{E_{Y'}})) = W'$ and this concludes the proof. □

Theorem 3.3. *Let X be a smooth, connected and projective scheme over W of relative dimension $dim(X/W) \geq 3$. Let H be a relatively very ample line bundle on X. Then there exists an integer d_1 (depending only on X) such that for any $d \geq d_1$, any $Y \in |H^{\otimes d}|$ connected and smooth over W and any section $y \in Y(W)$ the homomorphism $\widehat{\varphi} : \pi_1(Y_R, y_R) \to \pi_1(X_R, y_R)$ induced by the closed immersion $\varphi : Y \hookrightarrow X$ is an isomorphism.*

Proof. According to Theorem 3.2 we only need to prove that $\widehat{\varphi}$ is a closed immersion. This is equivalent to prove that for any finite extension K' of K and any essentially finite vector bundle V over Y' there exists an essentially finite vector bundle U over X' whose restriction $U_{|Y'} = \varphi'^*(U)$ is isomorphic to V (W', X' and Y' are constructed as in the proof of Theorem 3.2 and

$\varphi' : Y' \hookrightarrow X'$ is the morphism induced by φ). Let us set $X'_n := X' \times_W W/p^{n+1}$ and $Y'_n := Y' \times_W W/p^{n+1}$ for every nonnegative integer n, thus in particular $X'_0 = X'_k$ and $Y'_0 = Y'_k$. Let us denote by V_n the n-th restriction of V to Y'_n and similarly $\varphi'_n : Y'_n \hookrightarrow X'_n$ the n-th restriction of φ'. Now consider the cartesian diagram

$$\begin{array}{ccc} Y'_1 & \hookrightarrow & X'_1 \\ \uparrow & & \uparrow \\ Y'_0 & \hookrightarrow & X'_0. \end{array}$$

By [1] and [10] we know that there exists n_1 (depending only on X'_0) such that for every $d \geq n_1$ and for every $Y'_0 \in |H_k^{\otimes d}|$ the homomorphism

$$\widehat{\varphi_0} : \pi_1(Y'_0, y'_k) \to \pi_1(X'_0, y'_k)$$

between the fundamental group schemes of the special fibers of Y' and X', induced by $\varphi'_0 : Y'_0 \to X'_0$, is an isomorphism. This implies that there exists an essentially finite vector bundle U_0 over X'_0 such that ${\varphi'}^*_0(U_0) \simeq V_0$. From the exact sequence

$$0 \to \mathcal{O}_{X'_0}(-d) \to \mathcal{O}_{X'_0} \to \mathcal{O}_{Y'_0} \to 0$$

we obtain, first tensoring by $\mathcal{E}nd(U_0)$, the long exact sequence

$$H^1(X'_0, \mathcal{E}nd(U_0)(-d)) \longrightarrow H^1(X'_0, \mathcal{E}nd(U_0)) \xrightarrow{\delta_1} H^1(Y'_0, \mathcal{E}nd(V_0)) \longrightarrow$$

$$H^2(X'_0, \mathcal{E}nd(U_0)(-d)) \longrightarrow H^2(X'_0, \mathcal{E}nd(U_0)) \xrightarrow{\delta_2} H^2(Y'_0, \mathcal{E}nd(V_0)) \longrightarrow \cdots$$

By Lemmas 2.3 and 2.8 there exists a uniform positive integer d_1 such that $H^1(X'_0, \mathcal{E}nd(U_0)(-d))$ and $H^2(X'_0, \mathcal{E}nd(U_0)(-d))$ vanish for every $d \geq d_1$ thus in particular δ_1 is an isomorphism and δ_2 is injective. This implies that the induced maps:

$$\delta'_1 : H^1(X'_0, \mathcal{E}nd(U_0)) \otimes_k (p/p^2) \to H^1(Y'_0, \mathcal{E}nd(V_0)) \otimes_k (p/p^2)$$

and

$$\delta'_2 : H^2(X'_0, \mathcal{E}nd(U_0)) \otimes_k (p/p^2) \to H^2(Y'_0, \mathcal{E}nd(V_0)) \otimes_k (p/p^2)$$

are respectively an isomorphism and an injection. Now $X'_0 \to X'_1$ is a thickening of order one (cf. [7], (8.1.3)) so the obstruction for the existence of a vector bundle over X'_1 whose restriction over X'_0 is isomorphic to U_0 corresponds to an element $o(U_0)$ of $H^2(X'_0, \mathcal{E}nd(U_0)) \otimes_k (p/p^2)$ (cf. [7], Theorem 8.5.3 or [6], §6, Proposition 3). Then consider $\delta'_2(o(U_0)) = o(V_0) \in H^2(Y'_0, \mathcal{E}nd(V_0)) \otimes_k (p/p^2)$.

This is zero because clearly V_1 extends V_0, but δ'_2 is injective hence $o(U_0) = 0$ too and U_0 can thus be extended to a vector bundle U'_1 over X'_1. In general ${\varphi'}^*_1(U'_1)$ is not isomorphic to V_1, but we know (loc. cit.) that the set of deformations of V_0 over Y'_1 is an affine space under $H^1(Y'_0, \mathcal{E}nd(V_0)) \otimes_k (p/p^2)$. So there exists a unique $t \in H^1(Y'_0, \mathcal{E}nd(V_0)) \otimes_k (p/p^2)$ such that ${\varphi'}^*_1(U'_1) = V_1 + t$, then the vector bundle $U_1 := U'_1 - (\delta'_1)^{-1}(t)$ over X'_1 is such that ${\varphi'}^*_1(U_1) \simeq V_1$ over Y'_1 and of course U_1 is still a deformation of U_0 as $(\delta'_1)^{-1}(t) \in H^1(X'_0, \mathcal{E}nd(U_0)) \otimes_k (p/p^2)$ and the set of deformations of U_0 over X'_1 is an affine space under $H^1(X'_0, \mathcal{E}nd(U_0)) \otimes_k (p/p^2)$. Now consider the cartesian diagram

$$\begin{array}{ccc} Y'_2 & \hookrightarrow & X'_2 \\ \uparrow & & \uparrow \\ Y'_1 & \hookrightarrow & X'_1. \end{array}$$

The obstruction for the existence of a vector bundle U_2 over X'_2 deforming U_1 corresponds to an element $o(U_0)$ of $H^2(X'_0, \mathcal{E}nd(U_0)) \otimes_k (p^2/p^3)$; thus proceeding as in previous step we can find U_2 such that its restriction to Y'_2 is isomorphic to V_2. It is now clear that for any n we can construct a vector bundle U_n over X'_n extending U_{n-1} over X'_{n-1} whose restriction to Y'_n is isomorphic to V_n. Set $\widetilde{X'} := \underrightarrow{lim}_{i\in\mathbb{N}} X'_i$ and $\widetilde{U} := \underleftarrow{lim}_{i\in\mathbb{N}} U_i$. Then $\widetilde{U}$ is a vector bundle over $\widetilde{X'}$ and by [5] §5.1, we finally obtain a vector bundle U over X' whose restriction to Y' is isomorphic to V and whose special fiber is isomorphic to U_0, by construction. It remains to prove that U is essentially finite and this will be clear once we will prove that its generic fiber $U_{K'}$ is essentially finite over $X'_{K'}$. Since $V_{K'}$ is essentially finite then there exists a principal G-bundle $f : T \to Y'_{K'}$, for G an étale finite K'-group scheme, such that $f^*(V_{K'}) \simeq \mathcal{O}_T^{\oplus r}$, where $r := rk(U_{K'})$. By Grothendieck-Lefschetz's theorem in characteristic 0 we know that there exists a principal G-bundle $f' : T' \to X'_{K'}$ over $X'_{K'}$ whose restriciton to $Y'_{K'}$ is isomorphic to $f : T \to Y'_{K'}$:

$$\begin{array}{ccc} T & \hookrightarrow & T' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y'_{K'} & \hookrightarrow & X'_{K'}. \end{array}$$

We want to prove that f' trivializes $U_{K'}$. So let us consider the closed immersion $T \hookrightarrow T'$ and the associated short exact sequence

$$0 \to \mathcal{O}_{T'}(-d) \to \mathcal{O}_{T'} \to \mathcal{O}_T \to 0$$

that we tensor by $f'^*(U_{K'})$, so that we obtain the following long exact sequence

$$.. \to H^0(T', f'^*(U_{K'})) \to H^0(T, \mathcal{O}_T)^{\oplus r} \to H^1(T', f'^*(U_{K'})(-d)) \to ..$$

If we prove that $H^1(T', f'^*(U_{K'})(-d)) = 0$ then we are done: first of all we observe that U_k has zero Chern classes as V_k has zero Chern classes. This implies that $U_{K'}$ has zero Chern classes too. Moreover $U_{K'}$ is semistable because U_k is. Furthermore $char(K') = 0$ then in particular $H^1(T', f'^*(U_{K'})(-d)) = 0$ vanishes as desired.

□

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