

Bott manifold and cohomological rigidity

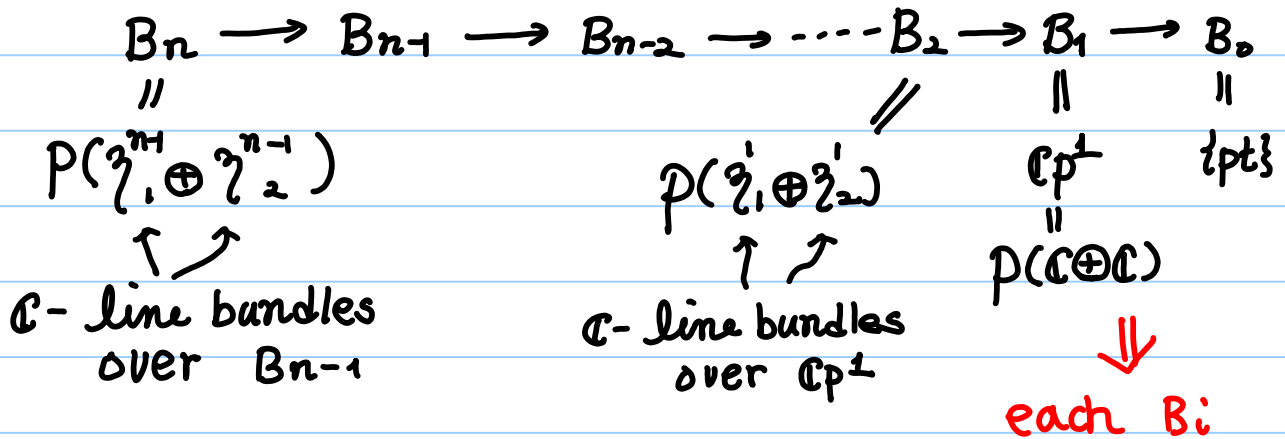
노트 제목

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호남수학회 위상수학 초청강연

1 Bott tower and Bott manifold.

Bott tower



In general

$$(1) P(\gamma_1 \oplus \gamma_2) \cong_{\text{diffeom}} P(\xi \otimes (\gamma_1 \oplus \gamma_2))$$

ξ \uparrow \mathbb{C} -line bundle

(2) Any complex line bundle ξ over M is determined by its 1-st Chern class $c_1(\xi) \in H^1(M; \mathbb{Z})$

$$\begin{aligned}
 \therefore P(\gamma_1 \oplus \gamma_2) &= P(\bar{\gamma}_1 \otimes (\gamma_1 \oplus \gamma_2)) \\
 &= P(\mathbb{C} \oplus \bar{\gamma}_1 \otimes \gamma_2)
 \end{aligned}$$

where $\bar{\gamma}_1$: \mathbb{C} -line bundle over M
 s.t. $c_1(\bar{\gamma}_1) = -c_1(\gamma_1)$.

\therefore We may consider $B_i = P(\mathbb{C} \oplus \xi_{i-1})$
 triv. bundle \nearrow line bundle / B_{i-1} .

2 Cohomology ring of Bott manifolds

$$\gamma_i: \underline{\mathbb{C}} \oplus \xi_{i-1} \xrightarrow{\tau_i} B_i = P(\underline{\mathbb{C}} \oplus \xi_{i-1})$$

canonical line bundle over B_i

$$x_i := c_1(\gamma_i) \in H^2(B_i)$$

\Rightarrow

$$\begin{aligned} H^*(B_i) &= H^*(B_{i-1})[x_i] / \begin{array}{l} x_i^2 = c_1(\xi_{i-1})x_i \\ \vdots \\ x_i^2 = c_1(\xi_i)x_i \end{array} \\ &= \mathbb{Z}[x_1, \dots, x_i] / \begin{array}{l} x_1^2 = 0 \\ x_2^2 = c_1(\xi_1)x_2 \\ \vdots \\ x_i^2 = c_1(\xi_i)x_i \end{array} \end{aligned}$$

Since $c_1(\xi_i) \in H^2(B_{i-1})$: generated by x_1, \dots, x_{i-1} ,
 $\Rightarrow c_1(\xi_i) = c_{i1}x_1 + \dots + c_{ii-1}x_{i-1}$
 for some $c_{ij} \in \mathbb{Z}$

\therefore The cohomology ring of $H^*(B_n)$

$$\cong \mathbb{Z}[x_1, \dots, x_n] / \begin{array}{l} \langle x_1^2, x_2(c_{21}x_1 + x_2), x_3(c_{31}x_1 + c_{32}x_2 + x_3) \\ \dots, x_n(c_{n1}x_1 + \dots + c_{nn-1}x_{n-1} + x_n) \rangle \end{array}$$

\leftrightarrow $\begin{bmatrix} 1 & 0 & \dots & 0 \\ c_{21} & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ c_{n1} & \dots & c_{nn-1} & 1 \end{bmatrix}$: associated matrix of the Bott tower.

3 Cohomological rigidity question for Bott manifolds

Question M, N : n -stage Bott manifolds
If $H^*(M; \mathbb{Z}) \cong H^*(N; \mathbb{Z})$ as graded rings
 $\stackrel{?}{\Rightarrow} M \cong N$ (homeomorphic or ?
diffeomorphic)

(If so, we say Bott manifolds are cohomologically rigid.)

[Masuda-Panov, 2008, Sbornik Math]

M : n -stage Bott manifold
If $H^*(M; \mathbb{Z}) \cong H^*((\mathbb{C}P)^n)$ trivial n -stage Bott manifold
 $\Rightarrow M \cong (\mathbb{C}P)^n$
diffeom

[Choi - Masuda - S 2009. To appear in Trans. AMS]

Any 3-stage Bott manifolds are cohomologically rigid,
i.e., M, N ; \cong
 $H^*(M) \cong H^*(N) \Rightarrow M \cong N$
diffeom

THEOREM 1

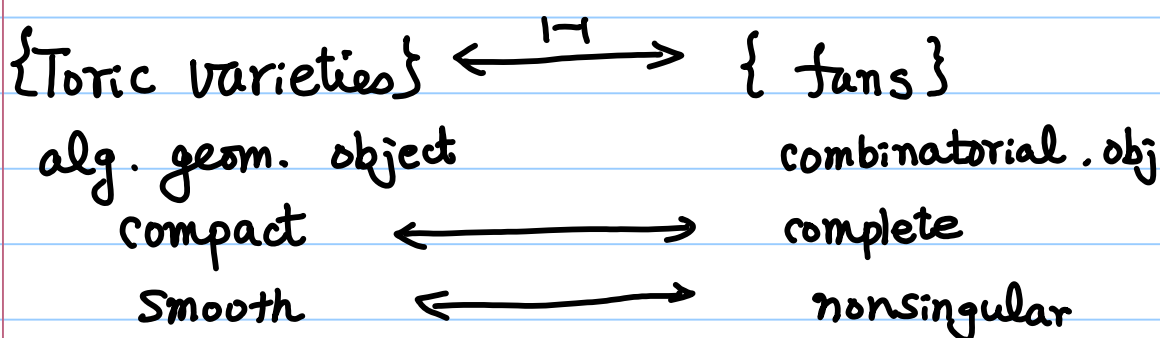
Any 1-twist Bott manifolds are cohomologically rigid

only one stage of the Bott tower sequence is a nontrivial fibration.

4. Toric manifold and quasitoric manifold.

Toric variety of dim n

= normal complex algebraic variety of dim n
with $(\mathbb{C}^*)^n$ -action having a dense orbit



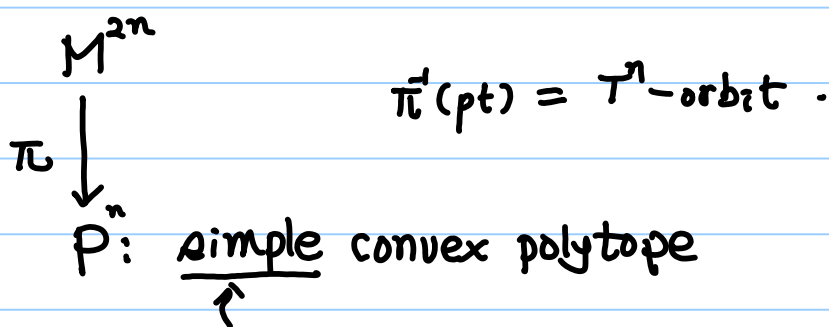
(E.g) (1) $\mathbb{C}P^n$: toric variety

$$(t_1, \dots, t_n) [z_0; z_1; \dots; z_n] = [z_0; t_1 z_1; \dots; t_n z_n]$$

(2) B_n : Bott mfd is a toric variety

assume B_{i-1} is a toric variety
 \exists $(\mathbb{C}^*)^{i-1}$ -action on B_{i-1} .
This action lifts to $(\mathbb{C}^*)^{i-1}$ -action on B_i
because $H^1(B_{i-1}) = 0$ (result of Yoshida-Hattori)
On the other hand there exists (not unique)
 (\mathbb{C}^*) -action on the line bundle η_{i-1}/B_{i-1} .
 \Rightarrow (\mathbb{C}^*) -action is induced on $p(\mathbb{C} \oplus \eta_{i-1}) = B_i$
 $\therefore \exists$ $(\mathbb{C}^*)^i$ -action on B_i is induced.

Quasitoric manifold (by Davis - Januszkiewicz)
 = closed $2n$ -dim. mfd M with
 locally standard $T^n = \underbrace{S^1 \times \dots \times S^1}$ action



at each vertex exactly n facets are
 intersecting

$\pi^{-1}(\text{vertex}) \in M^{T^n}$: fixed point set

$\pi^{-1}(\text{facet}) \subset$ fixed point set by a circle subgroup of T^n .

(E.g.) (1) $\mathbb{C}P^n$: g.t. manifold over Δ^n

(2) B^n : g.t. mfd over $(\Delta^1)^n = I^n$: n -cube.

(3) Any smooth toric varieties \Rightarrow g.t. mfd
 \Leftarrow

$\mathbb{C}P^1 \# \mathbb{C}P^1$: g.t. mfd over I^2

does not have almost complex structure

\therefore not a toric variety.

Question Is the class of smooth toric varieties
 (or g.t. mfd) cohomologically rigid?

5 Cohomological rigidity for g.t. mfd's which are cohomology Bott manifolds.

BQ-algebra of rank n over R

= graded algebra / R

generated by x_1, \dots, x_n of deg 2

$$\text{s.t. (i) } x_k^2 = \sum_{i < k} c_{ik} x_i x_k, \quad c_{ik} \in R$$

for $1 \leq k \leq n$

$$(2) \prod_{i=1}^n x_i \neq 0.$$

Cohomological complexity of BQ-algebra

$$= \# \text{ of } k\text{'s s.t. } x_k^2 \neq 0.$$

THEOREM 2.

M^{2^n}, N^{2^n} : g.t. manifolds s.t

s.t $H^*(M; \mathbb{Z}_{(2)}) \cong H^*(N; \mathbb{Z}_{(2)})$: BQ algebra / $\mathbb{Z}_{(2)}$

with cohomological complexity = 1.

Then $H^*(M; \mathbb{Z}) \cong H^*(N; \mathbb{Z}) \Rightarrow M \underset{\text{homeom}}{\approx} N$.

[\Rightarrow theorem is also true for \mathbb{Z} -coefficients]

THEOREM 3 M, N : 6-dim g.t. mfd

s.t $H^*(M; \mathbb{Z}) \cong H^*(N; \mathbb{Z})$: BQ-algebra / \mathbb{Z}

$\Rightarrow M \approx N$.

6 Sketch of proof of Theorem 2

We need the following theorems

Theorem 4. M^{2n} : g.t. mfd over P
if $H^*(M; \mathbb{Z}_{(2)})$ is a BQ-algebra / $\mathbb{Z}_{(2)}$.

$$\Rightarrow (1) P = I^n$$

(2) M is equivalent (hence homeom.)
to an n -stage Bott manifold.

Theorem 5. M : Bott manifold.

\Rightarrow twist number of a Bott tower structure
= cohomological complexity of $H^*(M; \mathbb{Z})$

Now let M, N : g.t. mfd at $H^*(M; \mathbb{Z}_{(2)})$ & $H^*(N; \mathbb{Z}_{(2)})$
are BQ-algebras of cohomological complexity = 1.

$$\Rightarrow \begin{array}{l} M \simeq B_n \rightarrow B_{n-1} \rightarrow \dots \rightarrow B_1 \rightarrow B_0: \text{ 1-twist} \\ N \simeq B'_n \rightarrow B'_{n-1} \rightarrow \dots \rightarrow B'_1 \rightarrow B'_0: \quad = \end{array}$$

\therefore By Theorem 1, $B_n \cong B'_n$

$\therefore M \simeq N$.

Proof of Theorem 1 requires some bundle theory
and more.