Bott manifold and cohomological rigidity

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1 Bott tower and Bott manifold.

Bott tower

$$\begin{array}{ccccccccccc}
B_n & \longrightarrow & B_{n-1} & \longrightarrow & B_{n-2} & \longrightarrow & \cdots & B_2 & \longrightarrow & B_1 & \longrightarrow & B_0 \\
\| & & & & & & & \| & & \| & & \| \\
P(\eta_1^{n-1} \oplus \eta_2^{n-1}) & & & & & & & P(\eta_1^1 \oplus \eta_2^1) & & \mathbb{C}P^1 & & \{pt\} \\
 & & & & & & & & & \| & & \\
 & & & & & & & & & P(\mathbb{C}\oplus\mathbb{C}) & &
\end{array}$$

$\mathbb{C}$- line bundles over B_{n-1}

$\mathbb{C}$- line bundles over $\mathbb{C}P^1$

$\Downarrow$

each B_i : i-stage Bott manifold.

In general

(1) $P(\eta_1 \oplus \eta_2) \underset{\text{diffeom}}{\cong} P(\xi \otimes (\eta_1 \oplus \eta_2))$

ξ : $\mathbb{C}$- line bundle

(2) Any complex line bundle ξ over M is determined by its 1-st Chern class $c_1(\xi) \in H^2(M;\mathbb{Z})$

$\therefore$ $P(\eta_1 \oplus \eta_2) = P(\bar{\eta}_1 \otimes (\eta_1 \oplus \eta_2))$

$= P(\mathbb{C} \oplus \bar{\eta}_1 \otimes \eta_2)$

where $\bar{\eta}_1$: $\mathbb{C}$- line bundle over M

s.t $c_1(\bar{\eta}_1) = -c_1(\eta_1)$.

$\therefore$ We may consider $B_i = P(\mathbb{C} \oplus \xi_{i-1})$

$\mathbb{C}$: trivial bundle; ξ_{i-1} : line bundle / B_{i-1}.

2 Cohomology ring of Bott manifolds

$$\gamma_i : \mathbb{C} \oplus \xi_{i-1} \xrightarrow{\pi_i} B_i = P(\mathbb{C} \oplus \xi_{i-1})$$

canonical line bundle over B_i

$$x_i := c_1(\gamma_i) \in H^2(B_i)$$

$\Rightarrow$

$$H^*(B_i) = H^*(B_{i-1})[x_i] \big/ x_i^2 = c_1(\xi_{i-1})x_i .$$

$$= \mathbb{Z}[x_1, \cdots x_i] \big/ \begin{array}{l} x_1^2 = 0 \\ x_2^2 = c_1(\xi_1)x_2 \\ \vdots \\ x_i^2 = c_1(\xi_i)x_i \end{array}$$

Since $c_1(\xi_i) \in H^2(B_{i-1})$: generated by $x_1, \cdots, x_{i-1}$.

$\Rightarrow$ $c_1(\xi_i) = c_{i1}x_1 + \cdots + c_{ii-1}x_{i-1}$

for some $c_{ij} \in \mathbb{Z}$

$\therefore$ The cohomology ring of $H^*(B_n)$

$$\cong \mathbb{Z}[x_1, \cdots, x_n] \big/ \langle x_1^2,\ x_2(c_{21}x_1 + x_2),\ x_3(c_{31}x_1 + c_{32}x_2 + x_3)\ \cdots\ x_n(c_{n1}x_1 + \cdots + c_{nn-1}x_{n-1} + x_n) \rangle$$

$$\longleftrightarrow \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ c_{21} & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ c_{n1} & \cdots & & c_{nn-1} & 1 \end{bmatrix}$$: associated matrix of the Bott tower.

3 Cohomological rigidity question for Bott manifolds

Question M, N : n-stage Bott manifolds

If $H^*(M;\mathbb{Z}) \cong H^*(N;\mathbb{Z})$ as graded rings

$\overset{?}{\Longrightarrow} M \cong N$ (homeomorphic or ? / diffeomorphic)

(If so, we say Bott manifolds are cohomologically rigid.)

[Masuda - Panov, 2008, Sbornik Math]

M : n-stage Bott manifold

If $H^*(M;\mathbb{Z}) \cong H^*((\mathbb{C}P^1)^n)$ trivial n-stage Bott manifold

$\Rightarrow M \underset{\text{diffeom}}{\cong} (\mathbb{C}P^1)^n$

[Choi - Masuda - S 2009. To appear in Trans. AMS]

Any 3-stage Bott manifolds are cohomologically rigid, i.e., M, N : "

$$H^*(M) \cong H^*(N) \Longrightarrow M \underset{\text{diffeom}}{\cong} N$$

only one stage of the Bott tower sequence is a nontrivial fibration.

THEOREM 1

Any 1-twist Bott manifolds are cohomologically rigid

" " .

4. Toric manifold and quasitoric manifold.

Toric variety of dim n
= normal complex algebraic variety of dim n
with $(\mathbb{C}^*)^n$- action having a dense orbit

$\{\text{Toric varieties}\} \xleftrightarrow{1-1} \{\text{fans}\}$

alg. geom. object — combinatorial. obj

compact $\longleftrightarrow$ complete

Smooth $\longleftrightarrow$ nonsingular

(E.g)(1) $\mathbb{C}P^n$: toric variety

$(t_1, \cdots, t_n)[z_0 : z_1 : \cdots : z_n] = [z_0 ; t_1 z_1 \cdots ; t_n z_n]$

(2) B_n : Bott mfd is a toric variety

assume B_{i-1} is a toric variety
$\exists$: $(\mathbb{C}^*)^{i-1}$- action on B_{i-1}.
This action lifts to $(\mathbb{C}^*)^{i-1}$-action on B_i
because $H^1(B_{i-1}) = 0$ (result of Yoshida-Hattori)
On the other hand there exists (not unique)
$(\mathbb{C}^*)$-action on the line bundle η_{i-1} / B_{i-1}.
$\Rightarrow$ $(\mathbb{C}^*)$-action is induced on $P(\mathbb{C} \oplus \eta_{i-1}) = B_i$
$\therefore$ $\exists$ $(\mathbb{C}^*)^i$-action on B_i is induced.

Quasitoric manifold (by Davis-Januszkiewicz)

= closed $2n$-dim. mfd M with locally standard $T^n = \underbrace{S^1 \times \cdots \times S^1}$ action

$$\begin{array}{ccc} M^{2n} & & \\ \pi \big\downarrow & & \pi^{-1}(pt) = T^n\text{-orbit}. \\ P^n & & \end{array}$$

P^n: simple convex polytope

at each vertex exactly n facets are intersecting

$\pi^{-1}(\text{vertex}) \in M^{T^n}$: fixed point set

$\pi^{-1}(\text{facet}) \subset$ fixed point set by a circle subgroup of T^n.

(E.g.) (1) $\mathbb{C}P^n$: q.t manifold over Δ^n

(2) B^n: q.t mfd over $(\Delta^1)^n = I^n$: n-cube.

(3) Any smooth toric varieties $\underset{\not\Leftarrow}{\Rightarrow}$ q.t. mfd

$\mathbb{C}P^1 \# \mathbb{C}P^1$; q.t. mfd over I^2

does not have almost complex structure

$\therefore$ not a toric variety.

Question Is the class of smooth toric varieties (or q.t mfds) cohomologically rigid?

5 Cohomological rigidity for q.t. mfds which are cohomology Bott manifolds.

BQ-algebra of rank n over R

= graded algebra / R

generated by $x_1, \dots, x_n$ of deg 2

s.t (1) $x_k^2 = \sum_{i<k} c_{ik} x_i x_k$, $c_{ik} \in R$

for $1 \le k \le n$

(2) $\prod_{i=1}^{n} x_i \neq 0$.

Cohomological complexity of BQ-algebra

= # of k's s.t $x_k^2 \neq 0$.

THEOREM 2.

M^{2n}, N^{2n} : q.t. manifolds s.t

st $H^*(M^{2n}; \mathbb{Z}_{(2)})$ & $H^*(N^{2n}; \mathbb{Z}_{(2)})$: BQ algebra / $\mathbb{Z}_{(2)}$

with cohomological complexity = 1.

Then $H^*(M;\mathbb{Z}) \cong H^*(M;\mathbb{Z}) \Rightarrow M \underset{\text{homeom}}{\approx} N$.

[$\Rightarrow$ theorem is also true for $\mathbb{Z}$-coefficients]

THEOREM 3 M, N : 6-dim q.t. mfd

st $H^*(M;\mathbb{Z}) \approx H^*(N;\mathbb{Z})$: BQ-algebra / $\mathbb{Z}$

$\Longrightarrow M \approx N$.

6 Sketch of proof of Theorem 2

We need the following theorems

Theorem 4. M^{2n}: q.t mfd over P

If $H^*(M; \mathbb{Z}_{(2)})$ is a BQ-algebra $/\mathbb{Z}_{(2)}$.

$\Rightarrow$ (1) $P = I^n$

(2) M is equivalent (hence homeom) to an n-stage Bott manifold.

Theorem 5. M: Bott manifold.

$\Rightarrow$ twist number of a Bott tower structure

$=$ cohomological complexity of $H^*(M; \mathbb{Z})$

Now let M, N: q.t. mfd s.t $H^*(M; \mathbb{Z}_{(2)})$ & $H^*(N; \mathbb{Z}_{(2)})$ are BQ-algebras of cohomological complexity $= 1$.

$\Rightarrow$ $M \approx B_n \to B_{n-1} \to \cdots \to B_1 \to B_0$: 1-twist

$N \approx B_n' \to B_{n-1}' \to \cdots \to B_1' \to B_0'$: "

$\therefore$ By Theorem 1, $B_n \cong B_n'$

$\therefore$ $M \approx N$.

Proof of Theorem 1 requires some bundle theory and more.